PERFORMANCE PRESERVING CONTROLLER ORDER REDUCTION FOR UNCERTAIN SYSTEMS

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Certificate

It is certified that the work contained in the thesis entitled "**Performance Preserving Controller Order Reduction for Uncertain Systems**", by **Debraj Chakraborty**, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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Abstract

μ-synthesis is a widely used controller design paradigm that utilizes the structure in the uncertainty associated with the plant. However the controller thus produced is typically of very high order. This dissertation is concerned with the approximation of the controller such that there is no degradation in the closed loop performance. A new proof has been proposed for the additive perturbation reduction technique with sufficient conditions to guarantee the closed loop structured singular value to remain less than unity. This proof provides a rigorous basis for controller reduction in the μ framework. A new coprime reduction technique has been proposed that makes unstable controller reduction possible with sufficient conditions to guarantee preservation of closed loop performance. The coprime factor perturbations to the controller have been shown to have a block diagonal structure. The proposed algorithms have been tested on a widely studied benchmark HIMAT aircraft and have been found to work satisfactorily producing more than 50% reduction in the controller order without optimization. Lastly, a full μ synthesis design has been performed on the experimental data available from a flexible launch vehicle. The details of the design with the selected performance weights and the simulation results are presented. The reduction algorithms are also tested on this practical example and found to produce considerable reduction in the controller order while preserving performance.

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Debraj.

To My Family.

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List of Symbols

Symbols	Meaning
$\mathbb{R}$	The real numbers
$\mathbb{C}$	The complex numbers
$\mathbb{R}^{p\times m}$	Real valued matrices with p rows and m columns
$\mathbb{C}^{p\times m}$	Complex valued matrices with p rows and m columns
M^*	The (complex conjugate) transpose of the matrix M
$\mathcal{S}(\cdot,\cdot)$	The Redheffer star product
$\lambda_i(M)$	The i^{th} eigenvalue of M
$\bar{\sigma}(M)$	The largest singular value of M
$\sigma_i(M)$	The i^{th} singular value of M
$M > 0$	$M = M^*$ is positive definite
$M \geq 0$	$M = M^*$ is positive semi-definite
$trace(M)$	The trace of M
$det(M)$	The determinant of M
$\rho(M)$	The spectral radius of M
$\mathcal{L}_2(j\mathbb{R})$	Square integrable functions on $\mathbb{C}$ including at ∞
$\mathcal{L}_\infty(j\mathbb{R})$	Functions bounded on $Re(s) = 0$ including at ∞
$\mathcal{H}_\infty$	The set of $\mathcal{L}_\infty(j\mathbb{R})$ functions analytic in $Re(s) > 0$
$\mathcal{RH}_\infty$	Rational functions in $\mathcal{H}_\infty$
$\mathcal{L}(\mathcal{V})$	Space of Bounded Operators from normed space $\mathcal{V}$ to $\mathcal{V}$
$G(s)$	Transfer function
$\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	Transfer function of the state space system (A, B, C, D)
$G^{\sim}(s)$	Shorthand for $G^T(-s)$
$\mathcal{F}_l(\cdot,\cdot)$	Lower LFT
$\mathcal{F}_u(\cdot,\cdot)$	Upper LFT

Introduction

The first step in controlling a system is to identify the system. Through identification a set of mathematical equations is derived that should accurately describe the dynamics of the system. However physical systems are very high dimensional, and an inherent tradeoff exists between the accuracy of the representation and the complexity of the equations chosen to represent it. In this thesis, we consider only systems that can be represented by finite dimensional linear time invariant differential equations. Thus the tradeoff mentioned above reduces to a minimization in the order of the differential equations that have been used to model the system without sacrificing the accuracy of such a representation. This so called model approximation problem for FDLTI systems has been extensively studied. One of the widely practised methods in this regard is to define a set of low dimensional systems, which, as a whole encompasses the complex behaviour of the real system. Thus this entire set is considered as a representation of the the original "uncertain" system. In other words this modelling procedure decreases complexity by introducing uncertainty into the system. The most common procedure to define this set of plants is to define a nominal plant and then consider a set of perturbations about this nominal plant to define the entire set. Depending on the structure of this set of perturbations different methods in describing the uncertainty have been evolved. This concept of uncertainty

is further extended so as to encompass other unknown or difficult-to-model parameters like nonlinearity and time variation etc. Under such a representation the control objective becomes designing a controller that provides stability and performance to all plants belonging to that set. Thus it is claimed that such a controller will be able to provide stability and performance when connected to the physical plant. Two of the most widely studied methods of controller design of such uncertain systems are the $\mathcal{H}_\infty$-synthesis and the μ-synthesis procedures. (See Francis [7], Doyle et al [5] and Packard and Doyle [24] for introductions to $\mathcal{H}_\infty$ and μ theory.)

Unfortunately the controllers designed by this paradigms are usually of very high order (typically equal to or greater than the order of the nominal plant). Practical controllers must be simple, linear and of low order. High complexity controllers are not only difficult to understand but they may turn out impossible to implement in hardware and software. Also from the point of integrity and reliability, simple controllers are preferable. In this dissertation we are chiefly concerned with the reduction of the state dimension of controller designed in a μ framework such that the closed loop structured singular value, with respect to the given uncertainty structure, remains less than unity. This guarantees that there shall be no degradation in the $\mathcal{H}_\infty$ performance of the closed loop system.

The design of reduced order controllers for high order plants may be divided into three categories [1] : direct design via optimization of controller parameters; model reduction followed by the design of a low order controller using the reduced order model; and design of a high order controller from the high order model, followed by reduction of the high order controller. The first method is evidently most desirable and much work has been done in this regard [14],[15],[16],[21]. However the problem

remains mathematically unsolved. The second method is intuitively appealing but it does not provide any a priori guarantee that the controller that is working for the low order model shall work for the actual system. In this dissertation we concentrate on the third method.

The main contribution of this thesis are as follows: a new proof for the Kavranoğlu's additive reduction technique [17] has been provided and the weights for two sided reduction have been derived. A new coprime factor reduction scheme has been proposed which guarantees closed loop stability and performance with structured uncertainty. The coprime factor perturbations to the controller have been found to have a block diagonal structure thus improving the reduction algorithm. In Chapter 4 a complete μ-synthesis design is performed for the experimental data available from a flexible launch vehicle.

The organization of the thesis is as follows:

Chapter 1 In this chapter a brief description of the mathematical tools needed in the analysis of the Controller Reduction problem are discussed. Concise outlines of the generic state space system, the Hardy Spaces, Internal Stability, Linear Fractional Transforms, Coprime and Spectral Factorizations, the Standard $\mathcal{H}_\infty$ suboptimal control problem and Uncertain Systems are given.

Chapter 2 Some of the current techniques for performance preserving controller reduction have been discussed in this chapter. The (P, γ)-admissible reduction technique proposed by Goddard and Glover [12] and modified by Wang et. al. [26] have been discussed in some details.

Chapter 3 The main results of this thesis are presented in this chapter.

Chapter 4 In this chapter the controller reduction techniques proposed in Chapter 3 has been tested on the experimental HIMAT aircraft and a flexible launch vehicle. The complete μ-synthesis design of the autopilot for the launch vehicle in also included.

Chapter 5 This concluding chapter summarizes the contributions of the thesis and points out the directions of possible further work in this problem.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter we briefly describe the mathematical tools and the relevant background needed in our analysis of the Controller Reduction problem. Section 1.2 covers the definition of the generic state space system while Section 1.3 defines the Hardy Spaces. The concepts of Internal Stability, Linear Fractional Transforms and Coprime and Spectral Factorizations are outlined in Section 1.4, 1.5 and 1.6 respectively. The Standard $\mathcal{H}_\infty$ suboptimal control problem is treated in Section 1.7 while Section 1.8 covers Uncertain Systems.

1.2 State Space Systems

In this dissertation we only consider plants which may be described by a set of linear, time invariant, finite dimensional, differential equations. Such systems may be represented by the following state space realization

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1.2.1}$$

where $x(t) \in \mathbb{R}^n$ is called the state vector, $u(t) \in \mathbb{R}^m$ is the input and $y(t) \in \mathbb{R}^p$ is the output. State matrices A, B, C, D are real, constant and of the appropriate dimensions. If $p = m = 1$ then the system is called scalar or SISO (single input single output) otherwise it is called multivariable or MIMO (multiple input multiple output).

If the Laplace transforms of $u(t)$ and $y(t)$ are $U(s)$ and $Y(s)$ respectively and assuming zero initial conditions, the transfer function matrix from input to output is defined as $G(s)$ where

$$Y(s) \;=\; G(s)U(s) \tag{1.2.2}$$

From this definition we have

$$G(s) \;=\; C(sI - A)^{-1}B + D \;=\; \left[\begin{array}{c|c} \mathrm{A} & \mathrm{B} \\ \hline \mathrm{C} & \mathrm{D} \end{array}\right] \tag{1.2.3}$$

where $\left[\begin{array}{c|c} \mathrm{A} & \mathrm{B} \\ \hline \mathrm{C} & \mathrm{D} \end{array}\right]$ is used as a shorthand notation for $G(s)$

Of all possible realizations (A, B, C, D) of G, those realizations with the smallest state dimension are called minimal. This is equivalent to (A, B) being controllable and (C, A) being observable. Given that (A, B, C, D) is minimal, the poles of G are defined as being the eigenvalues of A, and are closely related to the behaviour of the system. The system G is said to be stable if $Re(\lambda_i(A)) < 0$, and minimum phase if $Re(z) < 0$ where z is a transmission zero of the system.

Consider a stable transfer function G with realization (A, B, C, D). Then the controllability Gramian, P, and the observability Gramian, Q of G are defined as

$$\begin{aligned} P &= \int_0^\infty e^{At}BB^*e^{A^*t} \\ Q &= \int_0^\infty e^{A^*t}C^*Ce^{At} \end{aligned} \tag{1.2.4}$$

It can be verified by direct substitution that P and Q satisfy the following Lyapunov equations

$$\begin{aligned} AP + PA^* + BB^* &= 0 \\ A^*Q + QA + C^*C &= 0 \end{aligned} \tag{1.2.5}$$

Furthermore,(A, B) is controllable if and only if $P > 0$, and (C, A) is observable if and only if $Q > 0$.

The eigenvalues of the product PQ are invariant under similarity state transformations and hence form a set of input-output invariants.

Definition 1.2.1. Using the notation introduced above, the Hankel singular values of G are defined as

$$\sigma_i(G) = \sqrt{\lambda_i(PQ)}$$

The σ_i's can be reordered such that they are descending in magnitude, $\sigma_1 \geq ..... \geq \sigma_n$.

The largest singular value gives the worst case or maximum gain of the system denoted by G, and is widely used in Robust Control Literature. Detailed discussions on Linear Systems can be found in [2] and [4], while more on Hankel singular values can be found in [28].

1.3 The Hardy Spaces

For defining the $\mathcal{H}_\infty$ space we require the following Hilbert Spaces. A detailed treatment of the Hardy Spaces can be found in [6] and [28], while Hilbert Spaces have been covered in [13].

Definition 1.3.1. $\mathcal{L}_2(j\mathbb{R})$ or simply $\mathcal{L}_2$ is a Hilbert Space of matrix valued(or scalar-valued) matrix functions F on $j\mathbb{R}$ such that

$$\int_{-\infty}^{+\infty} trace[F^*(j\omega)F(j\omega)]d\omega < \infty$$

Definition 1.3.2. $\mathcal{H}_2$ is a (closed) subspace of $\mathcal{L}_2(j\mathbb{R})$with the matrix functions $F(s)$ analytic in $Re(s) > 0$.The corresponding norm is defined as

$$\|F\|_2^2 = \sup_{\sigma>0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} trace[F^*(\sigma + j\omega)F(\sigma + j\omega)]d\omega$$

Now we come to that class of functions bounded on the imaginary axis.

Definition 1.3.3. $\mathcal{L}_\infty(j\mathbb{R})$ or simply $\mathcal{L}_\infty$ is a Banach Space of matrix valued(or scalar-valued) matrix functions F that are (essentially) bounded on $j\mathbb{R}$,with norm

$$\|F\|_\infty = ess\sup_{\omega\in\mathbb{R}} \bar{\sigma}[F(j\omega)]$$

The rational subspace of $\mathcal{L}_\infty$, denoted $\mathcal{RL}_\infty$ consists of all proper rational transfer matrices with no poles on the imaginary axis.

Definition 1.3.4. $\mathcal{H}_\infty$ is a (closed) subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in $Re(s) > 0$ i.e. in the open right half plane with the norm defined as

$$\|F\|_\infty = \sup_{Re(s)>0} \bar{\sigma}[F(s)] = \sup_{\omega\in\mathbb{R}} \bar{\sigma}[F(j\omega)]$$

The $\mathcal{H}_\infty$ space along with this norm is used to characterize transfer functions in Robust Control Theory. The rational subspace of $\mathcal{H}_\infty$ denoted by $\mathcal{RH}_\infty$ consists of all proper and real rational stable transfer matrices.

1.4 Internal Stability

In the general feedback arrangement shown in figure 1.1, G and K are state space systems with the following equations

$$\dot{x}(t) = Ax(t) + [B_1\ B_2]\begin{bmatrix} w(t) \\ u(t) \end{bmatrix}, \tag{1.4.1}$$

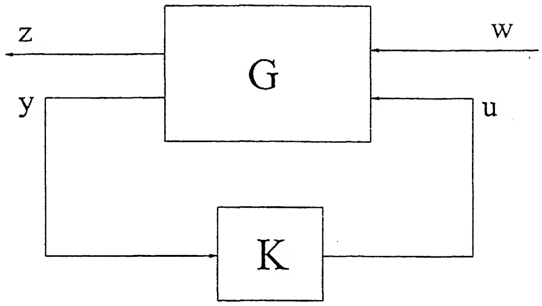

Figure 1.1: The Generic Feedback System

$$\begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x(t) + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \end{bmatrix} \tag{1.4.2}$$

and K being described by

$$\dot{x}_K(t) = A_K x_K(t) + B_K y(t), \tag{1.4.3}$$

$$u(t) = C_K x_K(t) + D_K y(t). \tag{1.4.4}$$

The system in Figure 1.1 is said to be *well-posed* if unique solutions exist for $x(t)$, $x_K(t)$, $y(t)$ and $u(t)$, for all initial conditions $x(0), x_K(0)$, and all input functions $w(t)$. Further such unique solutions must exist if the realizations are perturbed in some neighborhood of (A, B, C, D) and (A_K, B_K, C_K, D_K).

Theorem 1.4.1 (Proposition 5.1 in [6]). *The connection of G and K in Figure 1.1 is well-posed if and only if $I - D_{22}D_K$ is nonsingular.*

However well-posedness does not guarantee stability.

Definition 1.4.1. The system in Figure 1.1 is internally stable if it is well-posed, and for every initial condition $x(0)$ of G, and $x_K(0)$ of K, the limits $x(t), x_K(t) \overset{t\to\infty}{\longrightarrow} 0$ hold when $w = 0$.

Theorem 1.4.2 (Proposition 5.3 in [6]). *The system of Figure 1.1 is internally stable if and only if $I - D_{22}D_K$ is invertible and*

$$A_{cl} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix}$$

is Hurwitz.

1.5 Linear Fractional Transforms

This section defines the linear fractional transforms which shall be extensively used in this dissertation. See [28],[8],[27], and [6] for comprehensive treatments.

Definition 1.5.1. Let M be a complex matrix partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{C}^{(p_1+p_2)\times(q_1+q_2)}$$

and let $\Delta_l \in \mathbb{C}^{q_2\times p_2}$ and $\Delta_u \in \mathbb{C}^{q_1\times p_1}$ be two other complex matrices. Then we define a *lower LFT* with respect to Δ_l as the map

$$\mathcal{F}_l(M,\cdot) : \mathbb{C}^{q_2\times p_2} \mapsto \mathbb{C}^{p_1\times q_1}$$

with

$$\mathcal{F}_l(M,\Delta_l) := M_{11} + M_{12}\Delta_l(I - M_{22}\Delta_l)^{-1}M_{21}$$

provided that the inverse $(I - M_{22}\Delta_l)^{-1}$ exists. We can also define an *upper LFT* with respect to Δ_u as

$$\mathcal{F}_u(M,\cdot) : \mathbb{C}^{q_1\times p_1} \mapsto \mathbb{C}^{p_2\times q_2}$$

with

$$\mathcal{F}_u(M,\Delta_u) := M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12}$$

provided that the inverse $(I - M_{11}\Delta_u)^{-1}$ exists.

A useful interpretation of an LFT is that $\mathcal{F}_u(M, \Delta)$ has a nominal mapping M_{11} and is perturbed by Δ, while M_{12}, M_{21} and M_{22} reflect a prior knowledge as to how the perturbation affects the nominal map, M_{11}. This is why LFT is particularly useful in the study of perturbations, which is the focus of this dissertation..

1.6 Factorizations: Coprime and Spectral

In this section we briefly discuss the two type of factorization over $\mathcal{RH}_\infty$ which will be the key tools in our new method of controller reduction. Details of both Coprime and Spectral factorizations have been discussed in [7],[28] and [6].

1.6.1 Coprime Factorizations

Definition 1.6.1. Two matrices M and N belonging to $\mathcal{RH}_\infty$ are *right coprime over* $\mathcal{RH}_\infty$ if they have the same number of columns and if there exists matrices X_r and Y_r in $\mathcal{RH}_\infty$ such that

$$[X_r \ Y_r]\begin{bmatrix} M \\ N \end{bmatrix} = X_rM + Y_rN = I.$$

Similarly if two matrices $\tilde{M}$ and $\tilde{N}$ in $\mathcal{RH}_\infty$ are *left coprime over* $\mathcal{RH}_\infty$ if they have same number of rows and if there exists matrices X_l and Y_l in $\mathcal{RH}_\infty$ such that

$$[\tilde{M} \ \tilde{N}]\begin{bmatrix} X_l \\ Y_l \end{bmatrix} = \tilde{M}X_l + \tilde{N}Y_l = I.$$

These two equations are often called Bezout Identities.

Now let P be a proper real rational matrix. A *right coprime factorization (rcf)* of P is a factorization $P = NM^{-1}$, where N and M are right coprime over $\mathcal{RH}_\infty$. Similarly we can get a *left coprime factorization* of P.

Coprime factorizations are used extensively in both $\mathcal{H}_\infty$ synthesis and model approximation. A particular advantage of such multiplicative decompositions for model approximation is the stability of the two coprime factors.

1.6.2 Spectral Factorization

We require the following definition.

Definition 1.6.2. The conjugate transpose of $G(s)$ is denoted

$$G^{\sim}(s) = G(-s)^T = \left[\begin{array}{c|c} -A^* & C^* \\ \hline -B^* & D^* \end{array}\right]$$

Spectral factorizations may be viewed as the transfer function matrix equivalent of the square root. Given an Hermitian transfer matrix $\Phi = \Phi^{\sim}$ which has no poles on the $j\omega$ axis, spectral factorization allows us to calculate a stable, minimum phase spectral factor W such that $\Phi = W^{\sim}W$.

Theorem 1.6.1 (Chapter 7, Theorem 1, [7]). *Every square matrix $G(s)$ having the properties*

1. $G, G^{-1} \in \mathcal{R}L_\infty$

2. $G^{\sim} = G$

3. $G(\infty) > 0$

has a spectral factorization of the form $G = W^{\sim}W$ such that $W, W^{-1} \in \mathcal{RH}_\infty$.

A matrix satisfying the conditions of the above theorem also has a *co-spectral factorization* $G = WW^{\sim}$ such that $W, W^{-1} \in \mathcal{RH}_\infty$

1.6.3 Inner Functions

Inner transfer functions are used extensively in both $\mathcal{H}_\infty$ synthesis and model approximation.

Definition 1.6.3. A stable, $p \times m$ transfer function G, where $p \geq m$, is said to be inner if $G^{\sim}G = I$.

Definition 1.6.4. A stable, $p \times m$ transfer function G, where $p \geq m$, is said to be co-inner if $GG^{\sim} = I$.

The importance of inner functions stems from their norm preserving properties. If the input to an inner function has bounded energy then the output will be bounded by the same energy level.

1.7 The $\mathcal{H}_\infty$ Suboptimal Control Problem

In this dissertation we aim to design reduced order controllers from full order controllers such that the closed loop *structured singular value* remains acceptable. We use the so called μ-synthesis along with the $D - K$ Iteration to design the full order controller. The solution of the $\mathcal{H}_\infty$ suboptimal control problem, used as a part of the $D - K$ iteration is presented below. Consider the general feedback structure of figure 1.1. We use P here to denote the plant G to keep uniformity with usual nomenclature.

Here $P(s)$ given by

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{1.7.1}$$

where $B_1 \in \mathbb{R}^{n \times m_1}$, $B_2 \in \mathbb{R}^{n \times m_2}$, $C_1 \in \mathbb{R}^{p_1 \times n}$, $C_2 \in \mathbb{R}^{p_2 \times n}$, is a minimal state space realization of the generalized plant. The partition is assumed to be compatible with the signal dimensions. $P(s)$ includes both the actual plant to be controlled and any frequency dependent weights used in the controller synthesis process. It is assumed to be FDLTI.The two Riccati equation solution was proposed in [5], and a most extensive treatment can be found in [28]. For LMI solutions one can refer to [6]. Earlier results can be found in [7].

1.7.1 Problem Statement

Let γ be some pre-specified performance level.

Definition 1.7.1. A controller K is said to be (P, γ)-admissible if K stabilizes P and $\|T_{zw}\|_\infty = \|\mathcal{F}_l(P, K)\|_\infty < \gamma$.

The $\mathcal{H}_\infty$ suboptimal control problem is to find a K that is (P, γ)-admissible. However, the solution also provides a parameterization for all controllers that are (P, γ)-admissible.

The following assumptions are made for solving the problem:

1. (A, B_2, C_2, D_{22}) is stabilisable and detectable.

2. $D_{22} = 0$

3. $D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $D_{21} = [0 \ I]$ and $D_{11} = \begin{bmatrix} D_{1111} & D_{1112} \\ D_{1121} & D_{1122} \end{bmatrix}$,

   where $D_{1111} \in \mathbb{R}^{(p_1-m_2)\times(m_1-p_2)}$, $D_{1112} \in \mathbb{R}^{(p_1-m_2)\times p_2}$, $D_{1121} \in \mathbb{R}^{m_2\times(m_1-p_2)}$ and $D_{1122} \in \mathbb{R}^{m_2\times p_2}$

4. $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has a full column rank for all ω.

5. $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has a full row rank for all ω.

1.7.2 The Suboptimal Solution

Before stating the appropriate theorem it is convenient to define the following matrices:

$$R = D_1^* D_1 - \begin{pmatrix} \gamma^2 I & 0 \\ 0 & 0 \end{pmatrix} \ where \, D_1 = [D_{11} \ D_{12}]$$

and

$$\bar{R} = D_1 D_1^* - \begin{pmatrix} \gamma^2 I & 0 \\ 0 & 0 \end{pmatrix} \ where \, D_1 = \begin{bmatrix} D_{11} \\ D_{12} \end{bmatrix}$$

Also assuming solution exists, define X_∞ and Y_∞ as stabilizing solutions to the following Algebraic Riccati Equations,

$$\begin{aligned} (A - BR^{-1}D_1^*C_1)^* X_\infty + X_\infty(A - BR^{-1}D_1^*C_1) & \\ - X_\infty BR^{-1}B^* X_\infty + C_1^*(I - D_1R^{-1}D_1^*)C_1 &= 0 \end{aligned} \tag{1.7.2}$$

$$\begin{aligned} (A - B_1D_1^*\bar{R}^{-1}C)Y_\infty + Y_\infty(A - B_1D_1^*\bar{R}^{-1}C)^* & \\ - Y_\infty C^*\bar{R}^{-1}CY_\infty + B_1(I - D_1^*\bar{R}^{-1}D_1)B_1 &= 0 \end{aligned} \tag{1.7.3}$$

and F and H are state feedback and output-injection matrices respectively,

$$F = \begin{bmatrix} F_{11} \\ F_{12} \end{bmatrix} = -R^{-1}[D_1^* C_1 + B^* X_\infty],$$

$$\begin{aligned} H &= [H_{11} H_{12} H_2] \\ &= -[B_1 D_1^* + Y_\infty C^*]\tilde{R}^{-1}. \end{aligned}$$

We are now in a position to state necessary and sufficient conditions for the existence of (P, γ)-admissible controllers and a parameterization of all such controllers if any exists.

Theorem 1.7.1 (Theorem 1 in [12]). *For a plant described by equation 1.7.1 and satisfying assumptions 1-5:*

1. *There exists an internally stabilizing controller K such that $\|\mathcal{F}_l(P, K)\|_\infty < \gamma$ if and only if,*
   (i)$\gamma > max(\bar{\sigma}[D_{1111}\, D_{1112}], \bar{\sigma}[D_{1111}^* D_{1121}^*])$ and
   (ii)there exists $X_\infty \geq 0$ and $Y_\infty \geq 0$ satisfying equations (1.7.2) and (1.7.3) respectively such that $\rho(X_\infty Y_\infty) < \gamma^2$

2. *Given that the conditions of part (1) are satisfied, then all rational internally stabilizing controllers K satisfying $\|\mathcal{F}_l(P, K)\|_\infty < \gamma$ are given by the lower LFT $K = \mathcal{F}_l(K_a, \Phi)$, where $\Phi \in \mathcal{RH}_\infty$, such that $\|\Phi\|_\infty < \gamma$ and K_a is given by*

$$K_a = \begin{bmatrix} K_{a11} & K_{a12} \\ K_{a21} & K_{a22} \end{bmatrix} = \left[\begin{array}{c|cc} A_a & B_{a1} & B_{a2} \\ \hline C_{a1} & D_{a11} & D_{a12} \\ C_{a2} & D_{a21} & 0 \end{array}\right]$$

$$D_{a11} = -D_{1121} D_{1111}^* (\gamma^2 I - D_{1111} D_{1111}^*)^{-1} D_{1112} - D_{1122}$$

where $D_{a12} \in \mathbb{R}^{m_2 \times m_2}$ *and* $D_{a21} \in \mathbb{R}^{p_2 \times p_2}$ *are any matrices satisfying*

$$\begin{aligned} D_{a12}D_{a12}^* &= I - D_{1121}(\gamma^2 I - D_{1111}^* D_{1111})^{-1} D_{1121}^* \\ D_{a12}^* D_{a12} &= I - D_{1112}^*(\gamma^2 I - D_{1111} D_{1111}^*)^{-1} D_{1112} \end{aligned}$$

and

$$\begin{aligned} B_{a2} &= (B_2 + H_{12}) D_{a12} \\ C_{a2} &= -D_{a21}(C_2 + F_{12}) Z \\ B_{a1} &= -H_2 + B_{a2} D_{a12}^{-1} D_{a11} \\ C_{a1} &= F_2 Z + D_{a11} D_{a21}^{-1} C_{a2} \\ A_a &= A + HC + B_{a2} D_{a12}^{-1} C_{a1} \end{aligned}$$

where

$$Z = (I - \gamma^{-2} Y_\infty X_\infty)^{-1}.$$

Remark 1.7.1. D_{12a},D_{21a}, and hence K_a, are invertible.

Remark 1.7.2. $A_a - B_{2a} D_{12a}^{-1} C_{1a}$ and $A_a - B_{1a} D_{21a}^{-1} C_{2a}$ are both stable, hence $K_{a12}^{-1}, K_{a21}^{-1} \in \mathcal{H}_\infty$.

1.8 Uncertain Systems

In the last section we have given the primary result needed for synthesis of a suboptimal controller for a feedback system where the plant model was completely specified. Now we seek to address *uncertainty* in the plant arising out of various reason ranging from incomplete knowledge of the parameters to deliberate under modelling.

The idea is to parameterize the entire set of uncertain plants using perturbations about a nominal mapping.

We first present a motivating result from operator theory known widely as the Small Gain Theorem.

Theorem 1.8.1 (Small Gain Theorem(Theorem 3.7 in [6])). *Suppose Q is a member of the Banach algebra $\mathcal{B}$. If $\|Q\| < 1$, then $(I-Q)^{-1}$ exists. Furthermore*

$$(I-Q)^{-1} = \sum_{k=0}^{\infty} Q^k$$

1.8.1 The Structured Singular Value

We define a commonly encountered structure in our uncertainty class; Let S and F be nonnegative integers, not all zero, and let n, $s_1, ..., s_S$, $f_1, ..., f_F$ be positive integers, such that $n = \sum s_i + \sum f_i$. Consider the subspace of $n \times n$ complex matrices

$$C\mathbf{\Delta}_{S,F} = \{diag(\delta_1 I_{s_1}, ..., \delta_S I_{s_S}, \Delta_1, ..., \Delta_F) : \delta_i \in \mathbb{C} \text{ and } \Delta_i \in \mathbb{C}^{f_i \times f_i}\}$$

and denote its unit ball by $\mathbf{\Delta}_{S,F} = \{\Delta \in C\mathbf{\Delta}_{S,F} : \bar{\sigma}(\Delta) \leq 1\}$. Now define the time invariant uncertainty set as

$$\mathbf{\Delta}_{TI} = \{\Delta \in \mathcal{L}(\mathcal{L}_2) : \Delta \text{ is } LTI \text{ and } \Delta(s) \in \mathbf{\Delta}_{S,F}, \text{ for every } Re(s) \geq 0\}$$

Definition 1.8.1. Given a matrix $M \in \mathbb{C}^{n \times n}$, we define the structured singular value of M with respect to $\mathbf{\Delta}_{S,F}$ by

$$\mu(M, \mathbf{\Delta}_{S,F}) = \frac{1}{\min\{\bar{\sigma}(\Delta) : \Delta \in C\mathbf{\Delta}_{S,F} \text{ and } I - M\Delta \text{ is singular}\}}$$

when the minimum is defined. Otherwise $\mu(M, \mathbf{\Delta}_{S,F})$ is defined to be zero.

Computing μ has turned out to be extremely difficult and efficient algorithms are still the subject of intense research. We have to be satisfied with an upper bound. For that we define the set of matrices Θ which commute with the perturbations, that is, $\Theta\Delta = \Delta\Theta$ for all $\Delta \in \boldsymbol{\Delta}_{S,F}$. Define the set of positive matrices

$$\mathcal{P}\Theta_{\mathbf{S},\mathbf{F}} = \{diag(\Theta_1, ..., \Theta_S, \theta_1 I_{f_1}, ..., \theta_f I_{f_F}) : \Theta_i \in \mathbb{H}^{s_i}, \Theta_i > 0 \text{ } and \text{ } \theta_i > 0\}$$

where $\mathbb{H}$ denotes the set of Hermetian matrices.

Theorem 1.8.2 (Proposition 8.23 in [6]).

$$\mu(M, \boldsymbol{\Delta}_{S,F}) \leq \inf_{\Theta \in \mathcal{P}\Theta_{S,F}} \bar{\sigma}(\Theta^{\frac{1}{2}} M \Theta^{-\frac{1}{2}})$$

It can be shown that:

$$\begin{aligned} \mu(M, \boldsymbol{\Delta}_{TI}) &= \sup_{\omega \in \mathbb{R}} \mu(M(j\omega), \boldsymbol{\Delta}_{S,F}) \\ &\leq \sup_{\omega \in \mathbb{R}} \inf_{\Theta_\omega \in \mathcal{P}\Theta_{S,F}} \bar{\sigma}(\Theta^{\frac{1}{2}} M \Theta^{-\frac{1}{2}}) \end{aligned} \tag{1.8.1}$$

1.8.2 Feedback Control of Uncertain Systems

The generic feedback system is reprinted in Figure 1.2 but the perturbation Δ shown above incorporates the uncertainty in the plant while K represents the controller. The objective is to get such a K that will make the interconnection G and K internally stable which is equivalent to *nominal stability* of the interconnection. We also require that K should make the mapping

$$w \mapsto z = \mathcal{F}_u(\mathcal{F}_l(G, K), \Delta) = \mathcal{F}_u(M, \Delta)$$

small in some sense. Let $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ be partitioned compatibly with the input and output signals.

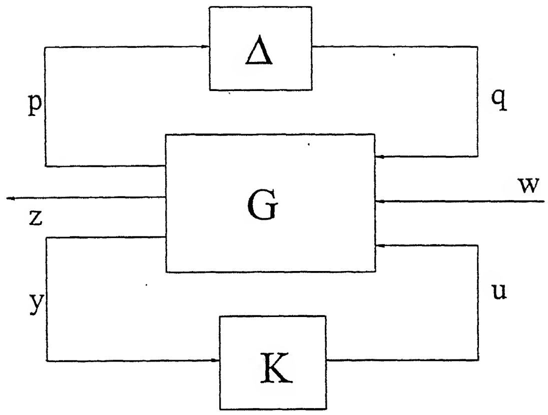

Figure 1.2: Feedback System with Uncertainty

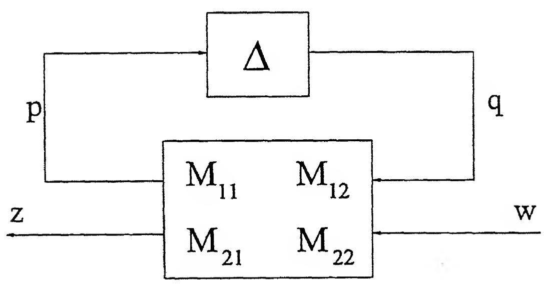

Figure 1.3: Equivalent Feedback System

Definition 1.8.2. With reference to figure 1.3 and with the extra assumption that the uncertainty class $\mathbf{\Delta}$ is causal, the uncertain system $(M_{11}, \mathbf{\Delta})$ is *robustly stable* if $(I - M_{11}\Delta)^{-1}$ exists in $\mathcal{L}(\mathcal{L}_2)$ and is causal for each $\Delta \in \mathbf{\Delta}$.

Definition 1.8.3. With reference to figure 1.3 and with the extra assumption that the uncertainty class $\mathbf{\Delta}$ is causal, the uncertain system $(M, \mathbf{\Delta})$ has *robust performance* if $(M_{11}, \mathbf{\Delta})$ is robustly stable and $\|\mathcal{F}_u(M, \Delta)\| < 1$, for every $\Delta \in \mathbf{\Delta}$.

The next theorem gives the major result of μ-synthesis providing us with a test of the μ norm to guarantee robust performance and stability.

Theorem 1.8.3 (Proposition 9.10 in [6]). *Suppose the configuration of Figure 1.3 is nominally stable, and the uncertainty class is $\mathbf{\Delta}_{TI}$ defined above. Define the perturbation set*

$$\mathbf{\Delta}_{TI,p} = \left\{ \begin{bmatrix} \Delta_u & 0 \\ 0 & \Delta_p \end{bmatrix} : \Delta_u \in \mathbf{\Delta}_{TI}, \Delta_p \in \mathcal{L}(\mathcal{L}_2)\ LTI,\ causal,\ \|\Delta_p\| \le 1 \right\}$$

Then the following are equivalent:

1. *The uncertain system $(M, \mathbf{\Delta}_{TI})$ satisfies robust performance: it is robustly stable and $\|\mathcal{F}_u(M, \Delta_u)\| < 1$, for every $\Delta_u \in \mathbf{\Delta}_{TI}$.*
2. *The uncertain system $(M, \mathbf{\Delta}_{TI,p})$ is robustly stable.*
3. $\sup_{\omega \in \mathbb{R}} \mu(M(j\omega), \mathbf{\Delta}_{S,F+1}) < 1$

We now shift our attention to the robust synthesis question. From the above discussion it can be shown that the robust synthesis for our standard setup in Figure 1.3 can be recast as the finding of the infimum of

$$\|\Theta \mathcal{F}_l(G, K) \Theta^{-1}\|$$

where K ranges over the stabilizing class $\mathcal{K}$, and Θ ranges over the set of rational scalings Θ_{TI} defined as

$$\Theta_{TI} = \{\Theta \in \mathcal{L}(\mathcal{L}_2) : \theta \textit{ nonsingular}, \ LTI \textit{ and } \Theta(s) = diag(\theta_1(s)I, ..., \theta_d(s)I)\}$$

The $D - K$ iteration is a widely used heuristic to perform the above minimization.The steps are as follows:

1. Set $\Theta_1 = I$, $\eta_0 = \infty$ and the counter k=1.

2. Solve for K_k in the $\mathcal{H}_\infty$ synthesis $\inf_{K_k} \|\Theta_k \mathcal{F}_l(G, K_k)\Theta_k^{-1}\|$;let η_k denote the achieved norm.

3. Compare η_k with η_{k+1};if they are approximately equal stop, and set $K = K_k$ as the final controller. Otherwise continue.

4. Solve for Θ_{k+1} in the scaled gain problem $\inf_{\Theta_{k+1}} \|\Theta_{k+1}\mathcal{F}_l(G, K_k)\Theta_{k+1}^{-1}\|$; increment k and go to step 2.

Chapter 2

Controller Reduction: Performance Preserving Approaches

2.1 Introduction

The most widely used controller design techniques like $\mathcal{H}_\infty$ loop shaping and μ-synthesis typically produce very high order controllers. But practical controllers must be simple, linear and of low order. High complexity controllers are not only difficult to understand but they may turn out impossible to implement in hardware and software. Also from the point of integrity and reliability, simple controllers are preferable.

The problem of reducing complexity via reducing the state dimension of the controller has been the subject of extensive research for the last forty years. See [1] and [11] for a complete list of references. In this dissertation we treat the controller reduction problem in a μ-synthesis framework and derive a set of sufficient conditions for making the closed loop μ with the reduced order controller less than one.

The design of reduced order controllers for high order plants may be divided into three categories [1] : direct design via optimization of controller parameters; model reduction followed by the design of a low order controller using the reduced order model; and design of a high order controller from the high order model, followed by reduction

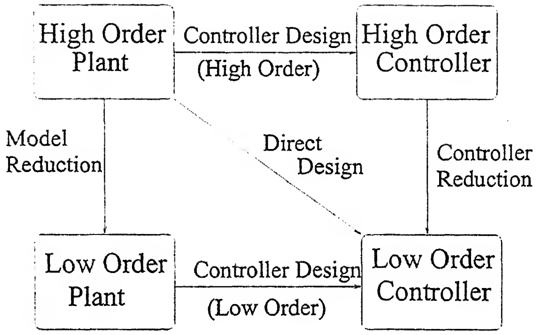

Figure 2.1: Different Paths to Controller Reduction

of the high order controller. These three paths are shown in Figure 2.1. Each of the three paths may be split into two categories: firstly those dealing with strictly open loop considerations - making no reference to how stability or other appropriate closed loop objectives may be affected by the use of such reduced order approximations; and secondly those techniques which look at controller approximation in a closed loop framework.

In this dissertation we will follow the third path and our goal will be to derive conditions that will guarantee closed loop stability and performance of the uncertain system with the reduced order controller. The approach will be to design a high order controller from a high order plant, followed by reduction of the high order controller. Information about the closed loop is incorporated into frequency weights and corresponds to requiring the modelling error to be smaller in selected frequency bands, e.g. around crossover. Hence the salient features of the high order controller are retained at the frequencies of most interest.

Several Open Loop Approximation techniques form the basis of recent advances in the controller reduction literature. Balanced Truncation, a widely used technique,

is based on the elimination of the least controllable and observable modes. The balancing of the gramians, proposed by Moore[23] has been used by Glover[9] among others to find an upper bound for the approximation error in terms of Hankel Singular Values.

The inability of this technique to provide any sort of optimal solution is overcome in the Optimal Hankel Norm Approximation Technique[9] which also provides a tighter upper bound in the approximation error than the balanced reduction.

However there is one common problem for these reduction schemes, that is, the model to be approximated is required to be stable. The coprime factorization approaches developed in [1],[20],[22] and [11],[12] have overcome this difficulty by decomposing the plant into two stable coprime factors, and then reducing the coprime factors instead of the plant.

As has been discussed in great details in [1],[11] and [12], controller reduction is a distinct problem from model reduction. But since standard solutions are available for the open loop approximation problems, it is of practical convenience to express the controller reduction problem as a model approximation problem. This is achieved in this dissertation through frequency weights.

This chapter is organized as follows: Section 2.1 discusses the three different paths to design a low order controller and mentions few of the open loop methods available in the literature. The frequency weighted Hankel norm approximation is covered in section 2.2 while sufficient conditions for stability preserving approximations are derived in section 2.3. The state of the art $\mathcal{H}_\infty$ performance preserving controller reduction techniques are detailed in section 2.4.

2.2 Frequency Weighted Approximation

The problem can be defined as:

$$\min_{\substack{\hat{K}\in\mathcal{RH}_\infty \\ deg(\hat{K})\leq deg(k)}} \left\|W_1(K-\hat{K})W_2\right\|_\infty \tag{2.2.1}$$

where $K \in \mathcal{RH}_\infty$, W_1 and W_2 are the weights chosen to preserve the salient features of the controller at the frequencies of interest. One common method is the Weighted Balanced Truncation[28], where the balancing can be done from the solution of two Lyapunov equations. But the drawback of this method is that there is no a priori bound on the achievable frequency weighted error. So we intend to use the frequency weighted optimal Hankel Norm Approximation which is presented next.

2.2.1 Frequency Weighted Optimal Hankel Norm Approximation

Since the design techniques to be presented in Chapter 3 lead to a two sided frequency weighted approximation problem with invertible weights, it is this framework which is now discussed.

To calculate a solution via the Nehari Extension Theorem[7],[28] we solve the equivalent problem

$$\inf_{\mathcal{Q}\in\mathcal{RH}^-_{\infty,(k)}} \left\|M_2^{\sim-1}(K-\mathcal{Q})M_1^{\sim-1}\right\|_\infty \tag{2.2.2}$$

where $M_1, M_1^{-1}, M_2, M_2^{-1} \in \mathcal{RH}_\infty$ such that $M_2^{\sim}M_2 = W_2W_2^{\sim}$ and $M_1M_1^{\sim} = W_1^{\sim}W_1$, and show that such a solution satisfies bounds on equation 2.2.1.

Define $H = [M_2^{\sim-1}KM_2^{\sim-1}]_+ \in \mathcal{RH}_\infty$. Then calculate an optimal Hankel norm approximation $\hat{H}$ of H,

$$\inf_{\hat{H}\in\mathcal{RH}^-_{\infty,(k)}} \left\|H-\hat{H}\right\|_\infty = \left\|H-\hat{H}\right\|_H = \sigma_{k+1}(H) \tag{2.2.3}$$

To calculate the required approximation to K notice that

$$\mathcal{Q} = M_2^{\sim}(\hat{H} + [M_2^{\sim -1} K M_1^{\sim -1}]_-) M_1^{\sim} \in \mathcal{RH}^-_{\infty,(k)}$$

Therefore setting $\hat{K} = [\mathcal{Q}]_+$, the stable part of $\mathcal{Q}$ gives a stable k^{th} order frequency weighted approximation to K.

The following lower bound or error for frequency weighted approximation can be derived:

$$\begin{aligned} \sigma_{k+1}([M_2^{\sim -1} K M_1^{\sim -1}]_+) &= \inf_{\mathcal{Q} \in \mathcal{RH}^-_{\infty,(k)}} \|W_2(K - \mathcal{Q})W_1\|_\infty \\ &\leq \inf_{deg(\hat{K}) < deg(K)} \left\|W_2(K - \hat{K})W_1\right\|_\infty . \end{aligned} \tag{2.2.4}$$

where $K \in \mathcal{RH}_\infty$ has McMillan degree n and $\hat{K} \in \mathcal{RH}_\infty$ has McMillan degree $k (k < n)$. Ideally we would like reduction techniques with an upper bound close to this lower bound.

An upper bound on the error for weighted Hankel norm approximation is much more difficult to obtain than for the unweighted case. Anderson[1] derives a bound for weighted Hankel norm approximations which is proportional to the unweighted error. The value of the factor of proportionality is dependent solely on the weighting functions. Glover et al[10] present the following error bound,

$$\left\|W_2(K - \hat{K})W_1\right\|_\infty \leq \sigma_{k+1}(H) + \left\|[\hat{H}]_- - [M_2^{\sim -1}(K - \hat{K})M_1^{\sim -1}]_-\right\|_\infty .$$

However, if either of the weights have zeros close to the imaginary axis then the second term can become arbitrarily large. Since this bound cannot be calculated a priori, (nor is it generally tight,) it is not as useful as the error bound in the unweighted case. For practical purposes, a combined model reduction and parameter optimization approach is used.

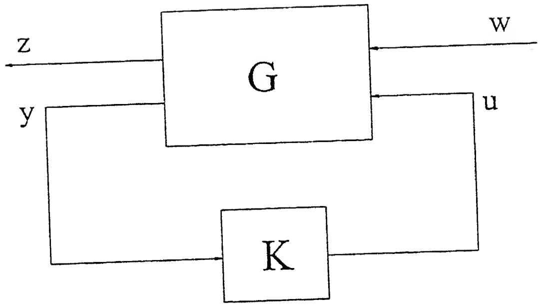

Figure 2.2: The generic Feedback System

Now we turn our attention to derive weights leading to open loop frequency weighted approximations, such that certain closed loop properties are retained. For a start we present the sufficient conditions for the weights to be stabilizing.

2.3 Stability Preserving Weights

We again consider the generic feedback system redrawn below in Figure 2.2. where G is given by

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

and $G_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ is a $p \times q$ transfer matrix. Suppose K is a m^{th} order controller that stabilizes the closed loop system. We are interested in investigating controller reduction methods that can preserve the closed-loop stability with the reduced order controllers. The following results are for the so called additive reduction.

Lemma 2.3.1 (Lemma 19.1 in [28]). *Let K be a stabilizing controller and $\hat{K}$ be a*

reduced order controller. Suppose $\hat{K}$ and K have the same number of right half plane poles and define

$$\Delta := \hat{K} - K, \; W_a := (I - G_{22}K)^{-1}G_{22}$$

Then the closed-loop system with $\hat{K}$ is stable if either

$$\|W_a\Delta\|_\infty < 1. \tag{2.3.1}$$

or

$$\|\Delta W_a\|_\infty < 1. \tag{2.3.2}$$

In view of the above lemma, the controller K should be reduced in such a way so that the weighted error $\left\|W_a(K - \hat{K})\right\|_\infty$ or $\left\|(K - \hat{K})W_a\right\|_\infty$ is small and K and $\hat{K}$ have the same number of unstable poles. If K is unstable, K is usually separated into stable and unstable parts as

$$K = K_+ + K_-$$

where K_+ is stable and only K_+ is reduced to $\hat{K}_+$, and the final reduced order controller is given by $\hat{K} = \hat{K}_+ + K_-$.

2.4 $\mathcal{H}_\infty$ Performance Preserving Techniques

In this section, we present a brief summary of current techniques od controller reduction preserving $\mathcal{H}_\infty$ performance. The assumptions of $\mathcal{H}_\infty$-suboptimal solution (Section 1.7.1) apply also to this section.

A major approach in this respect is the approximation in gap metric. However we choose to skip it here in order to concentrate on approaches which motivate our development in the μ framework. A bound on the performance degradation was

given by Lenz. et. al.[19]. When $K = K^0 + W_2 \Delta W_1$ is a nominal (P, γ) admissible controller subject to a weighted perturbation, they proposed a method to calculate weights such that

$$\left\| W_2^{-1}(K - K^0) W_1^{-1} \right\|_\infty < \frac{1}{\sqrt{2}} \Rightarrow \|\mathcal{F}_l(P, K)\|_\infty < \sqrt{2}. \tag{2.4.1}$$

Equation 2.4.1 represents a sufficient condition for controller K to stabilize the closed loop and give a performance bound of $\|\mathcal{F}_l(P, K)\|_\infty < \sqrt{2}$. This represents a (possible) degradation in performance of $\sqrt{2}$. A similar result may be derived by assuming K is a nominal controller subjected to coprime factor perturbations.

This bound was improved and also a practical algorithm for calculating the weights was proposed by Goddard and Glover in [11],[12].

2.4.1 (P, γ)-Admissible Controllers

As with the Lenz approach[19], in this case the sufficient conditions are expressed in terms of weighted error between a (reduced order) controller K and a (nominal) (P, γ)-admissible controller K^0, e.g.

$$\left\| W_2^{-1}(K - K^0) W_1^{-1} \right\|_\infty < 1 \Rightarrow \|\mathcal{F}_l(P, K)\|_\infty < \gamma. \tag{2.4.2}$$

For the coprime factor perturbation approaches, the sufficient conditions are expressed in terms of the weighted error between left coprime factors of a (reduced order) controller and a nominal (P, γ)-admissible controller. e.g.

$$\left\| W_2^{-1}([U \ V] - [U^0 \ V^0]) W_1^{-1} \right\|_\infty < 1 \Rightarrow \|\mathcal{F}_l(P, K)\|_\infty < \gamma. \tag{2.4.3}$$

where $K = V^{-1}U$ and $K^0 = {V^0}^{-1}U^0$. Equation 2.4.2 and 2.4.3 have a natural interpretation as frequency weighted model reduction problems. Given appropriate

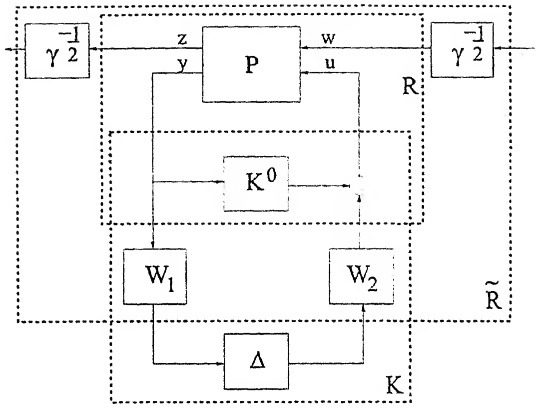

Figure 2.3: Plant and Controller Interconnection

weights, the goal is to calculate reduced order K or $[U\ V]$, which lies within an $\mathcal{L}_\infty$ ball (of right radius W_1 and left radius W_2), of K^0 or $[U^0\ V^0]$. K is then a reduced order (P, γ)-admissible controller. The larger the $\mathcal{L}_\infty$ ball easier the approximation. We therefore wish to make W_1 and W_2 large in some sense.

Assume that any controller K may be represented by a nominal admissible controller subjected to a weighted additive perturbation i.e. $K = K^0 + W_2 \Delta W_1$, where W_1 and W_2 are stable minimum phase and invertible and Δ is stable. This assumption imposes the restriction that K and K^0 possess the same right half plane poles, thus to some degree limiting the set of reduced order controllers attainable by this method.

Consider Figure 2.3. Observe that $\mathcal{F}_l(P, K) = \gamma \mathcal{F}_l(\tilde{R}, \Delta)$ and hence

$$\|\mathcal{F}_l(P, K)\|_\infty < \gamma \iff \left\|\mathcal{F}_l(\tilde{R}, \Delta)\right\|_\infty < 1 \qquad (2.4.4)$$

where

$$\tilde{R} = \begin{bmatrix} \gamma^{-\frac{1}{2}}I & 0 \\ 0 & W_1 \end{bmatrix} \mathcal{S}(P, \begin{bmatrix} K^0 & I \\ I & 0 \end{bmatrix}) \begin{bmatrix} \gamma^{-\frac{1}{2}}I & 0 \\ 0 & W_2 \end{bmatrix} \tag{2.4.5}$$

$\mathcal{S}(\cdot,\cdot)$ denotes the Redheffer star product. Suppose K^0 is some nominal (P, γ) admissible controller. If stable, minimum phase and invertible weights W_1 and W_2 can be calculated such that $\hat{R}$ is inner and $\hat{R}_{21}$ has full row rank (note that this is not generally possible-see Lenz et al [19] then

$$\begin{aligned} \|\Delta\|_\infty < 1 &\iff \|W_2^{-1}(K - K^0)W_1^{-1}q\|_\infty < 1 \\ &\iff \left\|\mathcal{F}_l(\tilde{R}, \Delta)\right\|_\infty < 1 \\ &\iff \|\mathcal{F}_l(P, K)\|_\infty < \gamma. \end{aligned} \tag{2.4.6}$$

Although it is not possible to calculate weights such that the above necessary and sufficient condition is satisfied, it is possible to calculate weights which make $\tilde{R}$ a contraction. Using such weights gives the sufficient condition,

$$\|W_2^{-1}(K - K^0)W_1^{-1}\|_\infty < 1 \Rightarrow \|\mathcal{F}_l(P, K)\|_\infty < \gamma. \tag{2.4.7}$$

Using similar methods the sufficient condition on the coprime factors as in equation 2.4.3 can be derived[12]. A method to calculate the weights so as to make $\tilde{R}$ a contraction is given in the following theorem.

Theorem 2.4.1 (Proposition 4.5.1 in [11]). *Assume* $\|R_{11}\| < \gamma$ *and define*

$$L = \begin{bmatrix} L_1 & L_2 \\ L_2^\sim & L_3 \end{bmatrix} = \mathcal{F}_l(\left[\begin{array}{cc|cc} 0 & -R_{22} & 0 & R_{21} \\ -R_{22}^\sim & 0 & R_{12}^\sim & 0 \\ \hline 0 & R_{12} & 0 & -R_{11} \\ R_{21}^\sim & 0 & -R_{11}^\sim & 0 \end{array}\right], \gamma^{-1}I). \tag{2.4.8}$$

Define $X = (W_1^{\sim} W_1)^{-1}$ *and* $Y = (W_2 W_2^{\sim})^{-1}$. *Then* $\tilde{R}$ *is a contraction if and only if*

$$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \geq \begin{bmatrix} L_1 & L_2 \\ L_2^{\sim} & L_3 \end{bmatrix}. \tag{2.4.9}$$

To obtain the set of controllers K such that $\|\Delta\|_\infty < 1$ is equivalent to finding the set of controllers K that lie within a weighted $\mathcal{L}_\infty$ ball of left radius W_2 and right radius W_1 centered on K^0. Hence we require both W_1 and W_2 to be large in some sense to make any subsequent approximation as easy as possible. In the scalar case maximizing $|W_1 W_2|$ is appropriate, however in the matrix case the optimal choice is not clear. Goddard and Glover [11],[12] have proposed maximizing the the product $trace(W_1^{\sim} W_1) trace(W_2 W_2^{\sim})$, and the product $det(W_1^{\sim} W_1) det(W_2 W_2^{\sim})$. We state only the relevant theorems giving the results of the respective optimizations.

Lemma 2.4.2 (Lemma 4.6.1 in [11]). *Define* L *as in Theorem 2.4.1 and assume* $L_1 \geq 0$, *with* $trace(L_1) \neq 0$,*and* $L_3 \geq 0$. *Also define*

$$\alpha_{opt} = \sqrt{(trace(L_3)/trace(L_1)}$$

$$Z_1^2 = L_2^{\sim} L_2$$

$$Z_2 = L_2 Z_1^{-1} L_2^{\sim}$$

where $Z_1 = Z_1^{\sim} > 0$ *and* $Z_2 = Z_2^{\sim} \geq 0$. *Then the weights* W_1 *and* W_2 *which maximize the product* $trace(W_1^{\sim} W_1) trace(W_2 W_2^{\sim})$ *subject to equation 2.4.9 satisfy*

$$(W_1^{\sim} W_1)^{-1} = L_1 + \alpha_{opt}^{-1} Z_2 \tag{2.4.10}$$

$$(W_2 W_2^{\sim})^{-1} = L_3 + \alpha_{opt} Z_1 \tag{2.4.11}$$

Lemma 2.4.3 (Lemma 4.6.2 in [11]). *Define L as in Theorem 2.4.1 and assume $L_1 \geq 0$ and $L_3 \geq 0$. Also define $SS^{\sim} = L_1 > 0$, $T^{\sim}T = L_3 > 0$,*

$$Z_1^2 = S^{-1}L_2L_3^{-1}L_2^{\sim}S^{\sim -1}$$

$$Z_2^2 = T^{\sim -1}L_2^{\sim}L_1^{-1}L_2T^{-1}$$

where $Z_1 = Z_1^{\sim} > 0$ and $Z_2 = Z_2^{\sim} \geq 0$. Then the weights W_1 and W_2 which maximize the product $det(W_1^{\sim}W_1)det(W_2W_2^{\sim})$ subject to equation 2.4.9 satisfy

$$(W_1^{\sim}W_1)^{-1} = S(I + Z_1)S^{\sim} = L_1 + SZ_1S^{\sim} \tag{2.4.12}$$

$$(W_2W_2^{\sim})^{-1} = T^{\sim}(I + Z_2)T = L_3 + T^{\sim}Z_2T. \tag{2.4.13}$$

A slightly modified approach has been proposed by Wang et al in [26], which gives lower order weights than the previous technique. Instead of using additive perturbation on the controller, this method is based on additive perturbation on the closed-loop transfer function.

Consider the class of closed-loop transfer functions with reduced order controller which can be represented as

$$G_r = G - W_2\Delta W_1$$

where G is the closed loop transfer functions with full order controller, Δ is stable perturbation, W_1 and W_2 are stable, invertible minimum phase weights. The following theorem gives the sufficient conditions for performance preserving reduction.

Theorem 2.4.4 (Theorem III.1 in [26]). *Assume $\|G\|_{\infty} < 1$, W_1 and W_2 are stable, invertible and minimum phase weighting functions and also satisfy*

$$\begin{bmatrix} I - W_2W_2^{\sim} & G \\ G^{\sim} & I - W_1^{\sim}W_1 \end{bmatrix} \geq 0 \tag{2.4.14}$$

$$\left\| W_2^{-1}(G - G_r)W_1^{-1} \right\|_\infty < 1 \tag{2.4.15}$$

Then

$$\|G_r\|_\infty < 1$$

Remark 2.4.1. From the above theorem it can be seen that $W_1^\sim W_1$ and $W_2^\sim W_2$ should satisfy the following inequalities:

$$I - W_2 W_2^\sim \geq G(I - W_1^\sim W_1)^{-1} G^\sim$$

$$I - W_1^\sim W_1 \geq G^\sim (I - W_2 W_2^\sim)^{-1} G$$

Since K_r is obtained by frequency weighted model reduction, it is preferable that W_1^{-1} and W_2^{-1} are as small possible in some sense. Instead of the maximizing $trace(W_1^\sim W_1) trace(W_2 W_2^\sim)$, Wang et al[26] chose to minimize

$$trace[W_2^{-1} G W_1^{-1} W_1^{-1\sim} G^\sim W_2^{-1\sim}]$$

The following theorem gives the sufficient conditions.

Theorem 2.4.5 (Theorem III.2 in [26]). *If* $\|G\|_\infty < 1$ *and*

$$\begin{bmatrix} I - W_2 W_2^\sim & G \\ G^\sim & I - W_1^\sim W - 1 \end{bmatrix} \geq 0 \tag{2.4.16}$$

then $\left\| W_2^{-1} G W_1^{-1} \right\|_2^2$ *reaches the minimum or* $trace[W_2^{-1} G W_1^{-1} W_1^{-1\sim} G^\sim W_2^{-1\sim}]$ *reaches the minimum with* $W_1^\sim W_1 = I - \sqrt{G^\sim G}$ *and* $W_2 W_2^\sim = I - \sqrt{G G^\sim}$

2.5 Conclusion

In this chapter we have reviewed some of the current techniques for performance preserving controller reduction. We have left out all the open loop approaches except

Hankel norm Approximation in order to emphasize on topics motivating the next chapter. The (P, γ)-admissible reduction technique proposed by Goddard and Glover and modified by Wang et. al. are especially important as they motivate our derivation of the sufficient conditions for performance preserving reduction in the μ framework to be presented in the next chapter.

Chapter 3

Controller Order Reduction of Uncertain Systems

3.1 Introduction

In the last chapter we reviewed several controller reduction techniques which guaranteed $\mathcal{H}_\infty$ performance of the closed loop system with the reduced order controller. These methods in general do not consider any structure in the uncertainty. However consideration of structure in the uncertainty may make the reduction schemes less conservative. Thus a greater reduction in the controller order may be possible. Moreover the proposed techniques integrate closely with the μ-synthesis controller design framework.

In this chapter we present two new controller reduction schemes which preserve the closed loop μ value to be less than one, thus guaranteeing closed loop robust stability and performance. Consideration of structure in the uncertainty is thus embedded in to the reduction algorithms and a generalization of current controller reduction schemes for the μ framework is achieved. Section 3.2 develops a new proof for Kavranoğlu's [17] additive reduction on the full order controller while 3.3 proposes a coprime factor reduction algorithm.

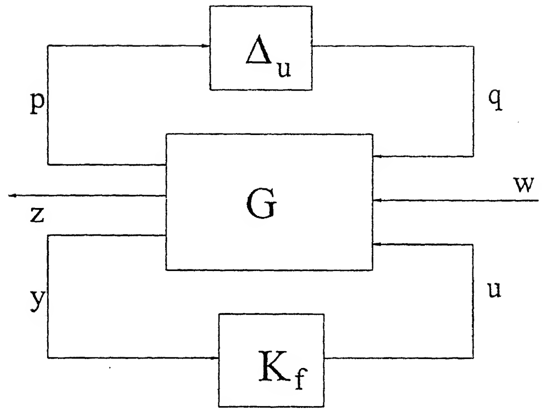

Figure 3.1: The Generic Feedback System with full order controller

3.2 Additive Perturbation Reduction

Let us again consider the generic uncertain feedback system of Figure 3.1. The uncertainty $\Delta_u \in \mathbf{\Delta}_{TI}$ as defined in Theorem 1.8.3. We use the same definitions of $C\mathbf{\Delta}_{S,F}$, $\mathbf{\Delta}_{TI}$ and $\mathbf{\Delta}_{TI,p}$ as in Section 1.8. Let K be a controller that makes the closed loop uncertain system $(\mathcal{F}_l(G,K), \Delta)$ robustly stable and having robust performance according to Definitions 1.8.2 and 1.8.3. Thus by Theorem 1.8.3

$$\mu(M, \mathbf{\Delta}_{TI,p}) = \sup_{\omega \in \mathbb{R}} \mu(M(j\omega), \mathbf{\Delta}_{S,F+1}) < 1 \tag{3.2.1}$$

where $M = \mathcal{F}_l(G,K)$. Conversely we can say, again from Theorem 1.8.3 that for any controller K, if we can show that $\mu(M(j\omega), \mathbf{\Delta}_{S,F+1}) < 1\ \forall \omega$, then the uncertain feedback system $(M, \mathbf{\Delta}_{TI})$ is robustly stable and has robust performance. This observation is the key to to the following controller reduction procedure.

Next we present a result that will be the basis of our controller reduction schemes. The main objective of the following result is to extend the standard mixed-μ analysis to the case when the complex uncertainties have frequency dependent upper bounds.

We augment the uncertainty structures to meet the mixed-μ setup. Let R, S and F be nonnegative integers, not all zero, and let n, $r_1, ..., r_R$, $s_1, ..., s_S$, $f_1, ..., f_F$ be positive integers, such that $n = \sum r_i + \sum s_i + \sum f_i$. The frequency dependent bounding set is thus defined as

$$\mathcal{W} = \{W = diag(\rho_1 I_{r_1}, ..., \rho_R I_{r_R}, w_1 I_{s_1}, ...w_S I_{s_S}, w_{S+1} I_{f_1}, ..., w_{S+F} I_{f_F})$$
$$: \rho_i > 0, i = 1, 2, ..., R, \ and\ some\ functions \quad (3.2.2)$$
$$w_i : \mathbb{R} \rightarrow \mathbb{R}_+, i = 1, ..., (S+F)\}$$

Let $C\Delta_{re}$ be the subspace of real $\sum r_i \times \sum r_i$ matrices defined by

$$C\Delta_{re} = \{diag(\gamma_1 I_{r_1}, ..., \gamma_R I_{r_R} : \gamma_i \in \mathbb{R}\} \quad (3.2.3)$$

Next we define a structure consisting of those functions that have continuous extension on the right half s-plane. Define the set:

$$C\Delta_{TI} = \{\Delta(s) \in \mathcal{H}_\infty : \Delta\ is\ LTI\ and\ \Delta(s) \in C\Delta_{S,F},\ for\ every\ Re(s) \geq 0\} \quad (3.2.4)$$

Next we define the augmented structure

$$\Delta_{rc} = \{\Delta_{rc} : \Delta_{rc} \in diag(\Delta_r, \Delta), \Delta_r \in C\Delta_{re}, \Delta \in C\Delta_{TI}\} \quad (3.2.5)$$

Finally we define the uncertainty set bounded above by the frequency dependent functions. Given $W \in \mathcal{W}$, define

$$\Delta_W = \{\Delta \in \Delta_{rc} : \Delta(j\omega)^* \Delta(j\omega) \leq W(\omega)^2,\ \forall \omega \in \mathbb{R}\}$$

Then we have the following result

Lemma 3.2.1 (Theorem 6 in [25]). *Let $W \in \mathcal{W}$ and $G \in \mathcal{H}_\infty$. If*

$$\mu(W(\omega)G(j\omega), \Delta_W) < 1,\ \forall \omega \in \mathbb{R},$$

then (G, Δ_W) is robustly stable.

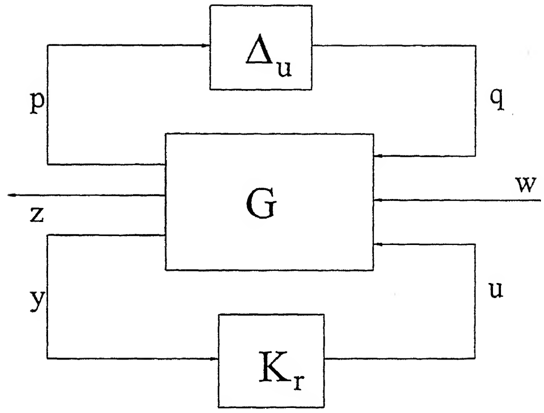

Figure 3.2: Uncertain system with the reduced order controller

We are now ready to give our main result for performance preserving controller reduction. To reframe the performance preserving controller reduction problem in the μ setting, consider the following block diagrams.(Figure 3.2 and 3.3). Let the reduced order controller K_r be expressed as a perturbation to the full order controller K_f such that $K_r = K_f + \Delta_c$. We assume that $\Delta_c \in \mathbf{\Delta}_W$. For the upper bound we define a frequency dependent function $W_a = \gamma(\omega)I$ where $\gamma(\omega) : \mathbb{R} \rightarrow \mathbb{R}_+$. Clearly $W_a \in \mathcal{W}$. Now define

$$T = \mathcal{S}(G, \begin{bmatrix} K_f & I \\ I & 0 \end{bmatrix}) \tag{3.2.6}$$

where $\mathcal{S}(\cdot,\cdot)$ denotes the Redheffer star product.

$$\tilde{T} = \begin{bmatrix} I & 0 \\ 0 & W_a \end{bmatrix} T \tag{3.2.7}$$

and

$$\mathbf{\Delta}_{upc} = \left\{ \begin{bmatrix} \Delta_{TI,p} & 0 \\ 0 & \Delta_c \end{bmatrix} : \Delta_{TI,p} \in \mathbf{\Delta}_{TI,p} \; and \; \Delta_c \in \mathbf{\Delta}_W \right\} \tag{3.2.8}$$

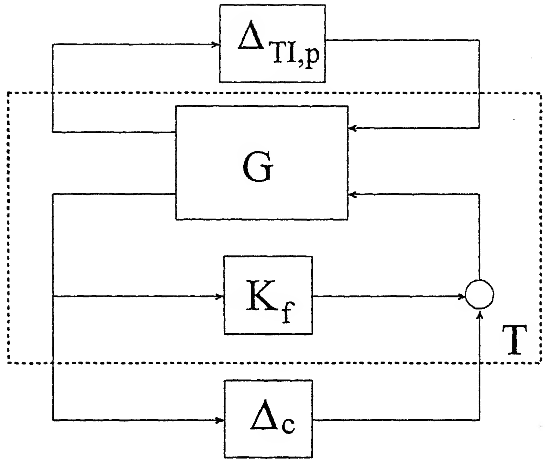

Figure 3.3: Plant and controller interconnection for additive reduction

A main drawback of this method is that this characterization of the set of reduced order controller limits $K_r(s)$ to have the same number of unstable poles as $K_f(s)$. This problem is addressed in the next section.

Proposition 3.2.2. *If* $\bar{\sigma}(\Delta_c(j\omega)) < \gamma(\omega)$, $\forall\omega$, *and* $\mu(\tilde{T}(j\omega), \Delta_{upc}) < 1$, $\forall\omega$, *then the closed loop uncertain system* $(\mathcal{F}_l(G, K_r), \Delta_{TI})$ *is robustly stable and has robust performance.*

Proof. Clearly $\begin{bmatrix} I & 0 \\ 0 & W_a \end{bmatrix} \in \mathcal{W}$ and by hypothesis for each ω

$$\bar{\sigma}(\Delta_c(j\omega)) < \gamma(\omega) \Rightarrow \; \Delta_c(j\omega)^*\Delta_c(j\omega) < W_a^2$$

$$\Rightarrow \Delta_{upc}(j\omega)^*\Delta_{upc}(j\omega) < \begin{bmatrix} I & 0 \\ 0 & W_a \end{bmatrix}^2$$

Now $\tilde{T} = \begin{bmatrix} I & 0 \\ 0 & W_a \end{bmatrix} T$. Also by hypothesis we have $\mu(\tilde{T}(j\omega), \Delta_{upc}) < 1$. Thus

by direct application of Lemma 3.2.1 we have the system $(T(j\omega), \Delta_{upc})$ is robustly stable. Now for any particular K_f and K_r we have

$$\begin{aligned}(T(j\omega), \Delta_{upc}) &\equiv (\mathcal{S}(G(j\omega), \begin{bmatrix} K_f & 0 \\ 0 & I \end{bmatrix}), \Delta_{upc}) \\ &\equiv (\mathcal{F}_l(G(j\omega), K_r), \Delta_{TI,p}).\end{aligned}$$

Thus we have shown that the closed loop uncertain system $(\mathcal{F}_l(G, K_r), \Delta_{TI,p})$ is robustly stable. But by theorem 1.8.3 this is equivalent to $(\mathcal{F}_l(G, K_r), \Delta_{TI})$ being robustly stable and having robust performance. Hence proved. □

Remark 3.2.1. From the above theorem it is evident according to Theorem 1.8.3 that if we can find an W_a such that $\mu(\tilde{T}(j\omega), \Delta_{upc}) < 1, \ \forall\omega$ then any K_r satisfying $\bar{\sigma}(K_f - K_r) < \gamma(\omega)$ is a robustly stabilizing controller and also provides robust performance to the closed loop system.

Now let $\hat{\gamma}(j\omega)$ be a rational function approximating $\gamma(\omega)$ such that $\hat{\gamma}(j\omega) < \gamma(\omega)$. Thus for each ω we have the following set of equivalences.

$$\begin{aligned}&\bar{\sigma}(K_f(j\omega) - K_r(j\omega)) < \hat{\gamma}(j\omega) \\ \iff &\tfrac{1}{\hat{\gamma}(j\omega)}\bar{\sigma}(K_f(j\omega) - K_r(j\omega)) < 1 \\ \iff &\bar{\sigma}[\tfrac{1}{\hat{\gamma}(j\omega)}I(K_f(j\omega) - K_r(j\omega))] < 1 \\ \iff &\bar{\sigma}[W_a^{-1}(K_f(j\omega) - K_r(j\omega))] < 1 \\ \iff &\left\|[W_a^{-1}(K_f(j\omega) - K_r(j\omega))]\right\|_\infty < 1\end{aligned}$$

Thus we have converted the controller reduction problem into the widely studied frequency-weighted $\mathcal{L}_\infty$ model approximation problem. This problem can be solved by a number of methods such as frequency weighted balanced truncation and optimal Hankel norm approximation.

Now the remaining problem is to calculate W_a such that $\mu(\tilde{T}, \Delta_{upc}) < 1$. This is done by the following technique. We have already assumed that $W_a(\omega) = \gamma(\omega)I$. We can find $\gamma(\omega)$ using among others the $\mu -$ *tools* software of MATLAB [8]. One has to perform a few bisection steps on the size of $\bar{\sigma}(K_f(j\omega) - K_r(j\omega))$ for each frequency to determine the maximum size possible such that $\mu(\tilde{T}(j\omega), \Delta_{upc}) = 1$.

Next we show that when expressed as a two sided frequency weighted approximation problem such as $\left\|W_1(K_f - K_r)W_2\right\|_\infty < 1$, the W_1 and W_2 produced by the above algorithm minimizes the weights in the sense that $trace(W_1^{\sim}W_1)trace(W_2W_2^{\sim})$ is minimized. For the next result we need the following standard lemma [26].

Lemma 3.2.3. *Suppose a Hermitian matrix is partitioned as* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ *where* $A = A^*$ *and* $C = C^{**} > 0$. *This matrix is positive-semidefinite if and only if* $A - BC^{-1}B^* \geq 0$.

We have the following proposition.

Proposition 3.2.4. *If* W_1 *and* W_2 *be two stable, invertible weighting function with minimum phase, satisfying* $W_1^{\sim}W_1 = \hat{\gamma}(j\omega)I$ *and* $W_2W_2^{\sim} = \hat{\gamma}(j\omega)I$ *then*

$$\left\|W_1(K_f - K_r)W_2\right\|_\infty < 1 \Leftrightarrow \bar{\sigma}(K_f(j\omega) - K_r(j\omega)) < \hat{\gamma}(j\omega)\ \forall\omega$$

and $trace(W_1^{\sim}W_1)trace(W_2W_2^{\sim})$ *is minimized.*

Proof. Let $M(s) = K_f(s) - K_r(s)$. We have

$$\begin{aligned}
&\|W_1 M W_2\| < 1 \\
&\Longleftrightarrow W_1 M W_2 W_2^{\sim} M^{\sim} W_1^{\sim} < I \\
&\Longleftrightarrow M W_2 W_2^{\sim} M^{\sim} < (W_1^{\sim} W_1)^{-1} \\
&\Longleftrightarrow \begin{bmatrix} (W_1^{\sim} W_1)^{-1} & M \\ M^{\sim} & (W_2 W_2^{\sim})^{-1} \end{bmatrix} > 0 \\
&\Longleftrightarrow \begin{bmatrix} 0 & M \\ M^{\sim} & 0 \end{bmatrix} > \begin{bmatrix} -(W_1^{\sim} W_1)^{-1} & 0 \\ 0 & -(W_2 W_2^{\sim})^{-1} \end{bmatrix} \\
&\Longleftrightarrow \begin{bmatrix} \hat{\gamma} I & M \\ M^{\sim} & \hat{\gamma} I \end{bmatrix} > \begin{bmatrix} \hat{\gamma} I - (W_1^{\sim} W_1)^{-1} & 0 \\ 0 & \hat{\gamma} I - (W_2 W_2^{\sim})^{-1} \end{bmatrix}
\end{aligned}$$

Now if $W_1^{\sim} W_1 > \hat{\gamma}(j\omega) I$ and $W_2 W_2^{\sim} > \hat{\gamma}(j\omega) I$, we have

$$\begin{aligned}
&\begin{bmatrix} \hat{\gamma} I - (W_1^{\sim} W_1)^{-1} & 0 \\ 0 & \hat{\gamma} I - (W_2 W_2^{\sim})^{-1} \end{bmatrix} > 0 \\
&\Longleftrightarrow \begin{bmatrix} \hat{\gamma} I & M \\ M^{\sim} & \hat{\gamma} I \end{bmatrix} > 0 \\
&\Longleftrightarrow M^{\sim} M < \hat{\gamma}^2 I
\end{aligned}$$

Therefore at each frequency

$$\begin{aligned}
&\bar{\sigma}(M(j\omega)) < \hat{\gamma}(j\omega) \\
&\Longleftrightarrow \bar{\sigma}(K_f(j\omega) - K_r(j\omega)) < \hat{\gamma}(j\omega) \ \forall \omega
\end{aligned}$$

Under this conditions $trace(W_1^{\sim} W_1) trace(W_2 W_2^{\sim})$ is trivially minimized for $W_1^{\sim} W_1 = W_2 W_2^{\sim} = \hat{\gamma}(j\omega) I$. Hence proved. $\square$

3.3 Coprime Factor Reduction

3.3.1 Introduction

One of the main drawbacks of the above method is that it assumes the controller to be approximated is stable. Robust controller design techniques like $\mathcal{H}_\infty$ loop shaping or μ-synthesis do not guarantee stability of the controllers. However it is possible to successfully approximate unstable systems. One simple method, if K is unstable, is to separate K into stable and unstable parts as $K = K_+ + K_-$ where K_+ is stable. Only K_+ is reduced to $\hat{K}_+$, and the final reduced order controller is given by $\hat{K} = \hat{K}_+ + K_-$. If K has $j\omega$ axis poles then they may also be copied directly into $\hat{K}$ but must have precisely the same residues.

The implication here is that no approximation can be undertaken on the unstable part of the controller. The reduced order controller is therefore constrained to possess the same unstable poles as the original controller. However, there is no reason why good approximations should possess this property.

An alternative to the above procedure is to consider a fractional description, the basis of which is multiplicative decomposition rather than additive decomposition. Fraction approaches to controller reduction have received much attention in the literature. These are generally termed as coprime factor approaches. Various sufficient conditions for the stability of the closed loop system when the controller is subjected to coprime factor perturbation has been derived in [20], [22]. $\mathcal{H}_\infty$ performance preserving techniques with multiplicative uncertainty have been studied in Goddard and Glover [12].

3.3.2 Coprime Factor Approach

In this section we propose a coprime factor based controller reduction scheme that keeps the closed loop μ value with the reduced order controller to remain less than one, thus guaranteeing robust stability of the closed loop system. Apart from having the natural advantages of an multiplicative scheme as mentioned in section 3.3.1, we also show that the perturbations have a block diagonal structure. This points to the possibility of greater reduction in the controller order than possible with the additive scheme described in Section 3.2.

We assume that the left coprime factors of $K_r = V_r^{-1}U_r$ may be represented by perturbations to the left coprime factors of the full order controller $K_f = V_f^{-1}U_f$ i.e.

$$U_r = U_f + \Delta_U \tag{3.3.1}$$

$$V_r = V_f + \Delta_V \tag{3.3.2}$$

We assume $\Delta_U, \Delta_V \in \mathbf{\Delta}_W$. Let $W_V = \gamma_V(\omega)I$ and $W_U = \gamma_U(\omega)I$ where $\gamma_V(\omega), \gamma_U(\omega) : \mathbb{R} \to \mathbb{R}_+$. We use the same definition of $\mathcal{W}$ as in Section 3.2. Clearly $W_V, W_U \in \mathcal{W}$. Figure 3.2 can be redrawn as in Figure 3.4, which after some manipulations give Figure 3.5. We define

$$K_{fA} = \begin{bmatrix} K_f & V_f^{-1} & V_f^{-1} \\ I & 0 & 0 \\ -K_f & -V_f^{-1} & -V_f^{-1} \end{bmatrix} \tag{3.3.3}$$

$$T = \mathcal{S}(G, K_{fA}) \tag{3.3.4}$$

and

$$\tilde{T} = \begin{bmatrix} I & 0 & 0 \\ 0 & W_V & 0 \\ 0 & 0 & W_U \end{bmatrix} T \tag{3.3.5}$$

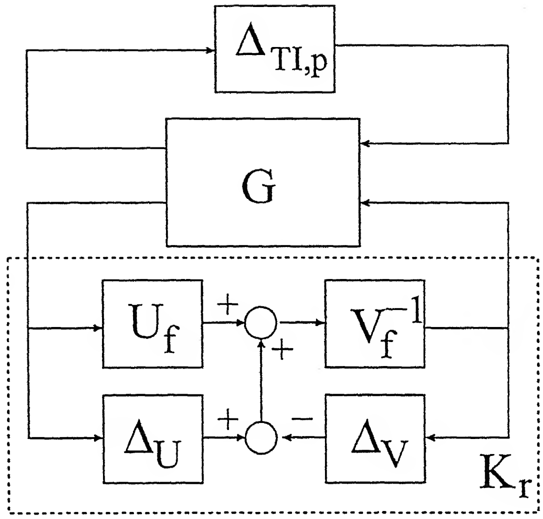

Figure 3.4: Uncertain system with reduced order controller

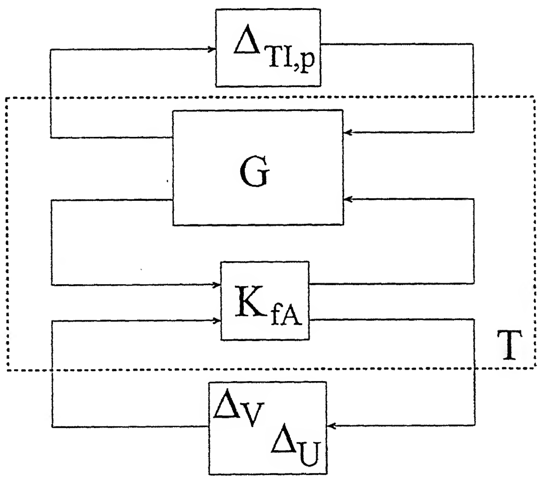

Figure 3.5: Plant and controller interconnection for coprime factor reduction

while

$$\mathbf{\Delta}_{upvu} = \left\{ \begin{bmatrix} \Delta_{TI,p} & 0 & 0 \\ 0 & \Delta_V & 0 \\ 0 & 0 & \Delta_U \end{bmatrix} : \Delta_{TI,p} \in \mathbf{\Delta}_{TI,p}, \Delta_V \in \mathbf{\Delta}_W, \Delta_U \in \mathbf{\Delta}_W \right\} \quad (3.3.6)$$

We have the following proposition to motivate our controller reduction procedure.

Proposition 3.3.1. *Let $\bar{\sigma}(\Delta_V(j\omega)) < \gamma_V(\omega)$ and $\bar{\sigma}(\Delta_U(j\omega)) < \gamma_U(\omega)$, $\forall\omega$. Now if $\mu(\tilde{T}(j\omega), \mathbf{\Delta}_{upvu}) < 1$, $\forall\omega$, then the closed loop uncertain system $(\mathcal{F}_l(G, K_r), \mathbf{\Delta}_{TI})$ is robustly stable and has robust performance.*

Proof. Clearly $\begin{bmatrix} I & 0 & 0 \\ 0 & W_V & 0 \\ 0 & 0 & W_U \end{bmatrix} \in \mathcal{W}$ and by hypothesis for each ω

$$\begin{aligned} & \bar{\sigma}(\Delta_V(j\omega)) < \gamma_V(\omega) \text{ and } \bar{\sigma}(\Delta_U(j\omega)) < \gamma_U(\omega) \\ \iff & \begin{bmatrix} \Delta_V & 0 \\ 0 & \Delta_U \end{bmatrix}^* \begin{bmatrix} \Delta_V & 0 \\ 0 & \Delta_U \end{bmatrix} < \begin{bmatrix} W_V & 0 \\ 0 & W_U \end{bmatrix}^2 \\ \iff & \Delta_{upvu}(j\omega)^* \Delta_{upvu}(j\omega) < \begin{bmatrix} I & 0 & 0 \\ 0 & W_V & 0 \\ 0 & 0 & W_U \end{bmatrix}_2^2 \end{aligned} \quad (3.3.7)$$

Now $\tilde{T} = \begin{bmatrix} I & 0 & 0 \\ 0 & W_V & 0 \\ 0 & 0 & W_U \end{bmatrix} T$. Also by hypothesis we have $\mu(\tilde{T}(j\omega), \mathbf{\Delta}_{upvu}) < 1$. Thus by direct application of Lemma 3.2.1 we have the system $(T(j\omega), \mathbf{\Delta}_{upvu})$ is robustly

stable. Now for any particular K_f and K_r we have

$$
\begin{aligned}
(T(j\omega), \Delta_{upvu}) &\equiv \left(\mathcal{S}\left(G(j\omega), \begin{bmatrix} K_f & V_f^{-1} & V_f^{-1} \\ I & 0 & 0 \\ -K_f & -V_f^{-1} & -V_f^{-1} \end{bmatrix} \right), \Delta_{upvu} \right) \\
&\equiv (\mathcal{F}_l(G(j\omega), K_r), \Delta_{TI,p}).
\end{aligned}
\tag{3.3.8}
$$

Thus we have shown that the closed loop uncertain system $(\mathcal{F}_l(G, K_r), \Delta_{TI,p})$ is robustly stable. But by theorem 1.8.3 this is equivalent to $(\mathcal{F}_l(G, K_r), \Delta_{TI})$ being robustly stable and having robust performance. Hence proved. □

Like in the additive case the above result implies that if we can find suitable $\gamma_V(\omega)$ and $\gamma_U(\omega)$ such that $\mu(\tilde{T}(j\omega), \Delta_{upvu}) < 1$, $\forall\omega$, then our controller reduction problem can be written as frequency weighted $\mathcal{L}_\infty$ approximation problems. Let $\hat{\gamma}_V(j\omega)$ and $\hat{\gamma}_U(j\omega)$ be two rational functions such that $\hat{W}_V(j\omega) = \hat{\gamma}_V(j\omega) < \gamma_V(\omega)$ and $\hat{W}_U(j\omega) = \hat{\gamma}_U(j\omega) < \gamma_U(\omega)\forall\omega$. Then our controller reduction problem reduces to the following set of frequency weighted $\mathcal{L}_\infty$ approximation problem following the arguments outlined in the additive case.

$$
\left\| W_V^{-1}(V_f - V_r) \right\|_\infty < 1 \tag{3.3.9}
$$

$$
\left\| W_U^{-1}(U_f - U_r) \right\|_\infty < 1 \tag{3.3.10}
$$

Thus our objective again reduces to finding $\gamma_V(\omega)$ and $\gamma_U(\omega)$ such that for each frequency $\mu(\tilde{T}(j\omega), \Delta_{upvu}) < 1$. Here also the bisection algorithm outlined in the additive case may be used. But search over two separate parameters is not practically possible as the dependance of μ on the bound of the uncertainty is not explicitly known. However we could just search over one parameter by taking $W_V = W_U = \gamma_{VU}(\omega)I$. Under this assumption we find out $\gamma_{VU}(\omega)$ by performing a few bisection steps on

the size of $\bar{\sigma}\left(\begin{bmatrix} \Delta_V(j\omega) & 0 \\ 0 & \Delta_U(j\omega) \end{bmatrix}\right)$ for each frequency to determine the maximum size possible such that $\mu(\tilde{T}(j\omega), \boldsymbol{\Delta}_{upvu}) = 1 \; \forall\omega$. The weighted $\mathcal{L}_\infty$ approximation problem can be solved by a number of available algorithms such as weighted balanced truncation or optimal Hankel norm approximation. In our design problems discussed in the next chapter we use the frequency weighted optimal Hankel norm approximation for its tighter upper bound of approximation error.

3.4 Summary

In this chapter two performance preserving techniques in the μ-framework has been proposed. A new proof has been given for the Kavranoğlu's additive reduction technique and the weights for two sided reduction have been derived. A new coprime factor reduction scheme has been proposed which guarantees closed loop stability and performance with structured uncertainty. The coprime factor perturbations to the controller have been found to have a block diagonal structure thus improving the reduction algorithm.

Chapter 4

Design Examples

4.1 Introduction

In this chapter we deal with the practical design aspects of μ-synthesis and controller order reduction techniques discussed in Chapter 3. The two numerical examples consider μ-synthesis of reduced order controllers for LTI systems. Given a generalized plant P and the uncertainty set Δ , the objective of μ-synthesis design is to find a controller K which stabilizes the closed loop uncertain system $\mathcal{F}_u(P, \Delta)$ and makes the norm $\|\mathcal{F}_u(\mathcal{F}_l(P, K), \Delta)\| < 1$. The generalized plant incorporates both the actual plant which is to be controlled and weighting functions which specify performance objectives. The performance weights are selected to emphasize objectives such as disturbance attenuation at low frequency and robustness to modelling uncertainty. Selection of appropriate weights is nontrivial and often involves a certain amount of iteration. For a more detailed discussion on weight selection see [8],[3] and [18].

In Section 4.2 we examine the application of the proposed algorithms on the pitch control of an experimental aircraft model. This example was chosen as it has been widely studied in the controller reduction literature and thus can be used for comparative purposes. Section 4.3 details the complete μ synthesis design and the

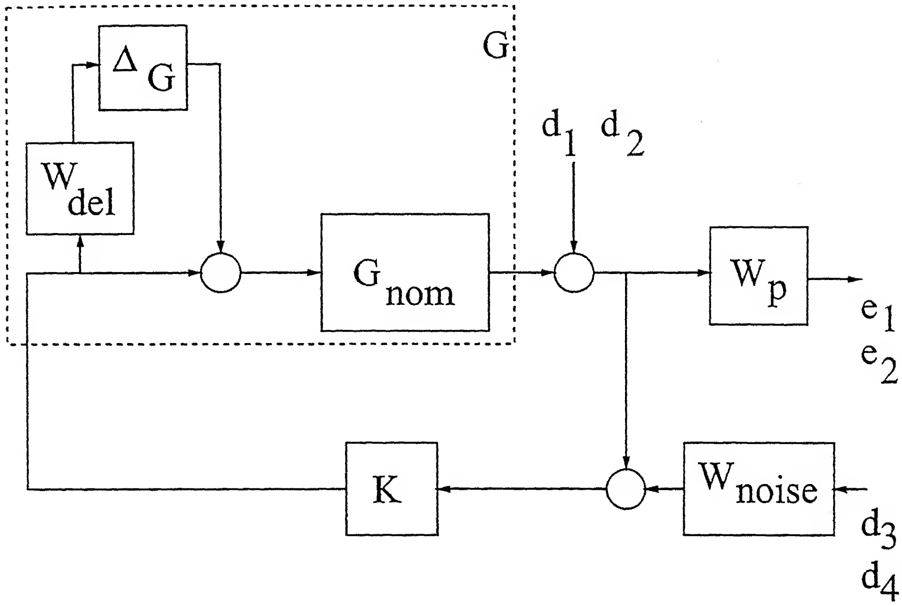

Figure 4.1: The HIMAT Closed Loop Interconnection

reduction of the full order controller for the experimental data available from a flexible Launch Vehicle.

4.2 Pitch Control of the HIMAT Model

4.2.1 The Plant Description

In this example the controller order reduction algorithms of Chapter 3 are applied to the design of a pitch axis controller of an experimental highly maneuverable airplane, namely HIMAT. A linearized model and detailed specifications can be found in [8].

A diagram for the closed loop system which includes the feedback structure of the plant and the controller, and elements associated with the uncertainty and performance specifications, is shown in figure 4.1. The actual aeroplane is represented by

the dashed box with the transfer function G, with the combination of W_{del} and Δ_G representing the amount and type of uncertainty in the nominal model G_{nom}. G_{nom} has four states

δv-perturbation along the velocity vector

α-angle of attack

q-pitch rate

θ-pitch angle

two control inputs,

δ_e-elevon command

δ_c-canard command

two measured outputs,

α-angle of attack

θ-pitch angle

The flight path angle is given by $\rho = \theta - \alpha$. In [8], a 30 state $\mathcal{H}_\infty$ controller is designed using μ-synthesis to give robust performance for the following three maneuvers:

Vertical Translation: Control the vertical velocity at constant θ (α varies). This implies that the attitude remains constant as the velocity vector rotates.

Pitch Pointing: Control the attitude at a constant flight path angle (i.e. $\theta - \alpha$). For this maneuver the velocity vector does not rotate.

Direct Lift: Control the flight path angle while holding the angle-of-attack constant (i.e. $\rho = \theta$). This maneuver produces a normal acceleration without changing the angle-of-attack.

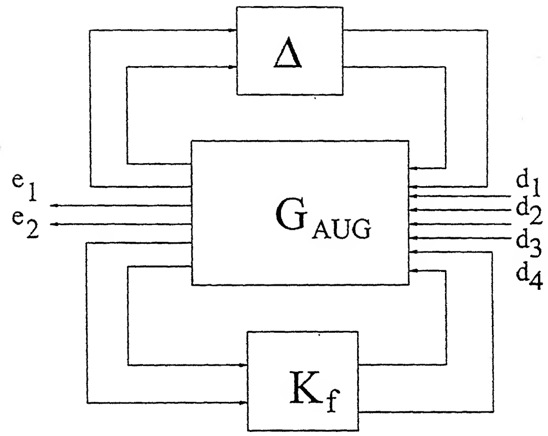

Figure 4.2: The Augmented Plant and Controller

These performance objectives are captured in the weight Wp. Thus the design objectives are translated into the requirement that the transfer functions from the disturbances $[d_1\, d_2\, d_3\, d_4]^T$ to $[e_1\, e_2]^T$ to be less than unity for all $||\Delta_G|| < 1$.

A generalized plant incorporating all the weights is first formed. The resulting plant G_{AUG} along with the uncertainty set Δ_G and the controller K_f is shown in figure 4.2.

4.2.2 Controller Reduction Results

The generalized plant G_{AUG} has 10 states. K_f is a 30 state controller satisfying the performance specifications. The closed loop robust performance is conveniently measured in terms of the μ of the closed loop with respect to an augmented uncertainty set. We define

$$\Delta_{AUG} := \left\{ \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} : \Delta_1 \in \mathbb{C}^{2\times 2}, \Delta_2 \in \mathbb{C}^{2\times 2} \right\}$$

Reduction Approach	Controller Order	$\mu(\mathcal{F}_l(P,K),\Delta)$
Full Order	30	0.966
$\mu A(P)$	16	1.0567
$\mu C(P)$	15	1.0536
$\mu A(P)$+B.T.	8	1.1219

Table 4.1: Results of Model Reduction

According to theorem 1.8.3 the Augmented plant G_{AUG} has robust performance and robust stability if $\mu(\mathcal{F}_l(G_{AUG},K_f),\Delta_{AUG}) < 1$. In this case K_f is found to satisfy $\mu(\mathcal{F}_l(G_{AUG},K_f),\Delta_{AUG}) \leq 0.996$

The results of model reduction are presented in Table 4.1. The following abbreviations are used

$\mu A(P)$ Additive Perturbation Reduction

$\mu C(P)$ Coprime Factor Perturbation Reduction

This can be compared with the results from [12] and [26]. Table II from [26] is reprinted below. In Table 4.2, $\mathcal{A}(\Phi)$, $\mathcal{CF}(\Phi)$ and $\mathcal{CF}(\Theta)$ are the methods proposed in [12]. UOHNA represents unweighted optimal Hankel norm Approximation, UBT represents unweighted balanced truncation and LCFR represents unweighted optimal Hankel norm reduction of controller left coprime factors.

Thus the proposed algorithms are found to reduce the 30^{th} order controller to comparable orders as in the table above. It should be noted that the full order controller used in Table 4.2 above is of order 20 while the proposed approaches in Table 4.1 uses a 30^{th} order full controller.

It is suspected that further reduction is possible but for the poor performance of the actual frequency weighted reduction algorithms. In fact a 8^{th} order controller

Reduction Approach	Controller Order	$\|\mathcal{F}_l(P,K)\|_\infty$
Full Order	20	0.97
$\mathcal{A}(\Phi)$	14	0.99
$\mathcal{CF}(\Phi)$	16	0.98
$\mathcal{CF}(\Theta)$	16	0.99
UOHNA	13	0.98
UBT	13	0.98
LCFR	13	0.98
$\mathcal{A}(\Phi)$	7	1.27
$\mathcal{CF}(\Phi)$	7	7.07
$\mathcal{CF}(\Theta)$	10	2.02

Table 4.2: Results from [26]

can be found simply by balanced truncation of the reduced controller of the $\mu A(P)$ approach, which satisfies the sufficient condition i.e. $\mu(\mathcal{F}_l(G_{AUG}, K_r), \Delta_{AUG}) < 1$.

It should further be noted that the main advantage of the proposed methods is not utilized in this example as the original uncertainty of the HIMAT system was unstructured. With the incorporation of structure in the uncertainty the proposed methods remain applicable. This is not the case with most of the currently available techniques which do not incorporate any knowledge of the uncertainty structure.

Figures 4.3, 4.4 and 4.5 show the closed loop μ plots for the full order and the reduced order controllers obtained by each of the proposed algorithms. The peak value 0f 0.966 for the full order controller increases to 1.05 for the reduced order controllers. However for further reduction the peak μ value increases to unacceptable high values. So 16 with $\mu A(P)$ and 15 with $\mu C(P)$ are the minimum orders achievable with the proposed reduction techniques.

The closed loop singular value plots are shown in figure 4.6, which clearly shows that all the plots are well below unity. This indicates successful $\mathcal{H}_\infty$ design.

Figures 4.7, 4.8 and 4.9 show the time responses (for the first two seconds) for a

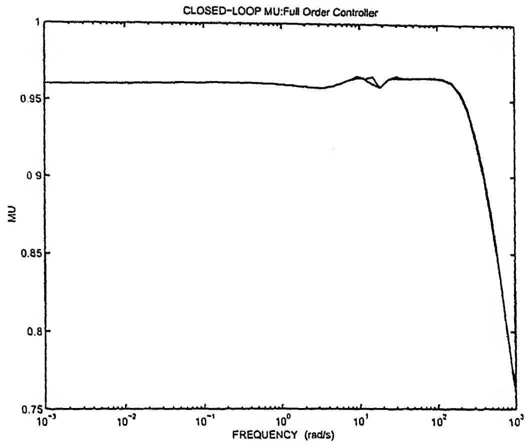

Figure 4.3: The HIMAT Closed Loop μ with Full Order Controller

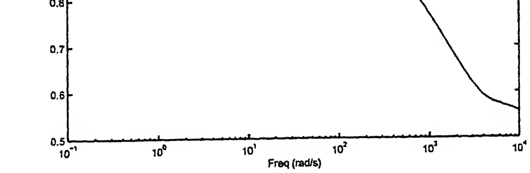

Figure 4.4: The HIMAT Closed Loop μ with Reduced Order($\mu A(P)$) Controller

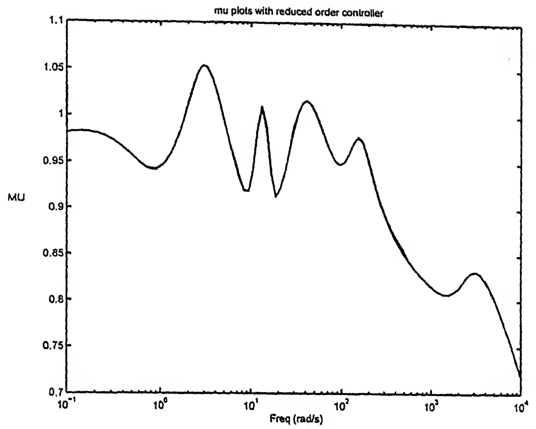

Figure 4.5: The HIMAT Closed Loop μ with Reduced Order($\mu C(P)$) Controller

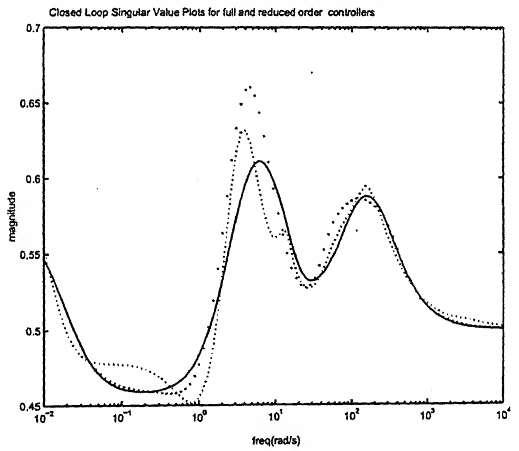

Figure 4.6: The Closed Loop Singular Values:continuous-full order; dashed-$\mu A(P)$; dotted-$\mu C(P)$

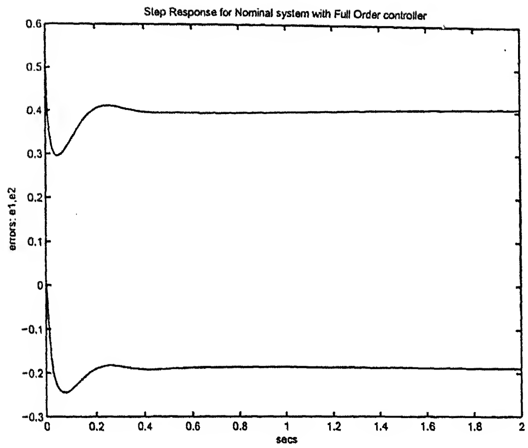

Figure 4.7: Step Response with Full Order Controller for Nominal System

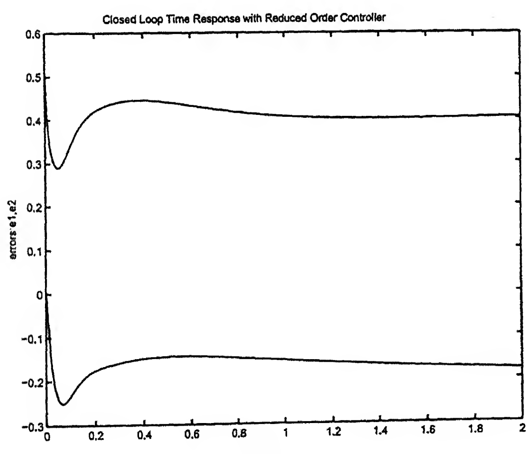

Figure 4.8: Step Response with $\mu A(P)$ Controller for Nominal System

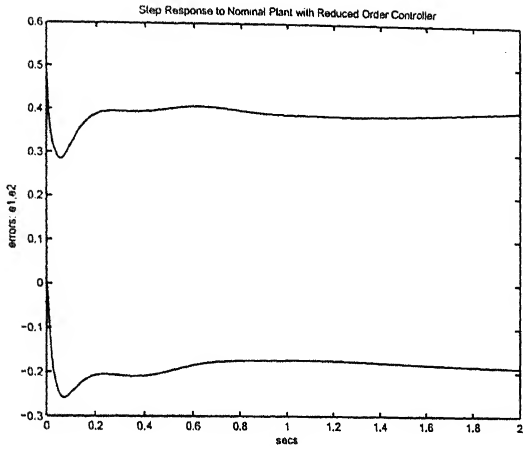

Figure 4.9: Step Response with $\mu C(P)$ Controller for Nominal System

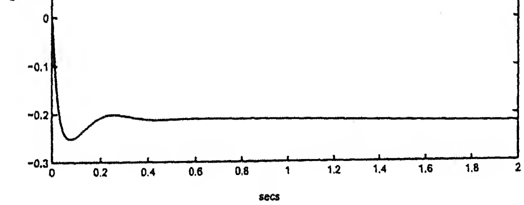

Figure 4.10: Step Response with Full Order Controller for Perturbed System

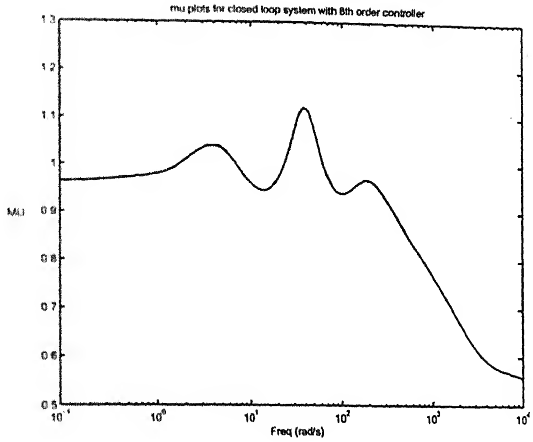

Figure 4.13: The HIMAT Closed Loop μ with 8th Order Controller

step input into the first channel, for the nominal plant. The responses are very similar with slight oscillations in the reduced order plots which are quickly damped out. This again indicates that the reduction approaches are successful. Time responses for the same input signal for the plant perturbed by $\Delta_G = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.1 \end{bmatrix}$ is shown in figures 4.10, 4.11 and 4.12. These figures show that the reduced order controller performs as well as the full order controller even for the perturbed plants.

The corresponding results for the optimized 8^{th} order controller are presented through the figures 4.13 to 4.16. While the closed loop peak μ has shot up to just above 1.1, the singular value plots are still well below unity. The step responses are as good as the full order controller for the nominal as well as the perturbed plant. Thus the proposed techniques have successfully achieved a reduction in the controller order from 30 to 8 preserving robust stability and performance.

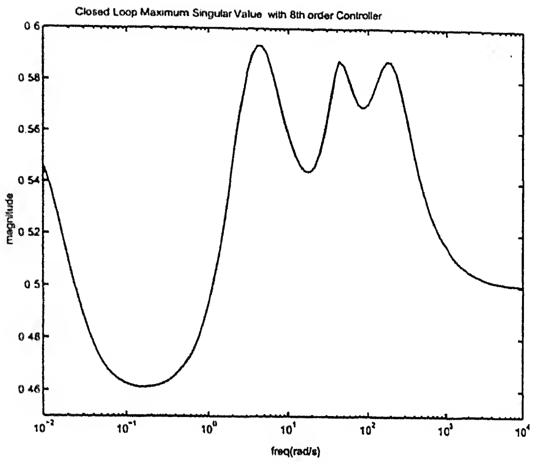

Figure 4.14: The Closed Loop Singular Value with 8th Order Controller

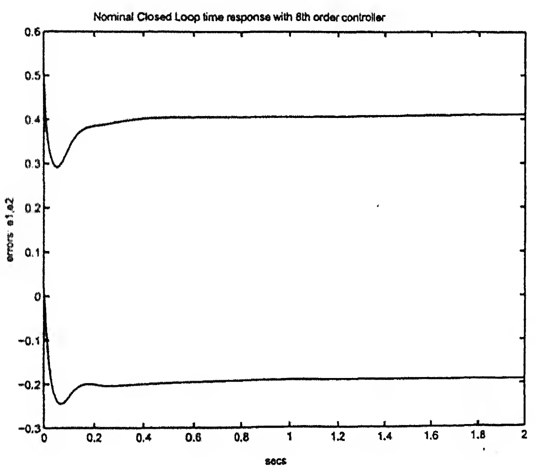

Figure 4.15: Step Response with 8th Order Controller for Nominal System

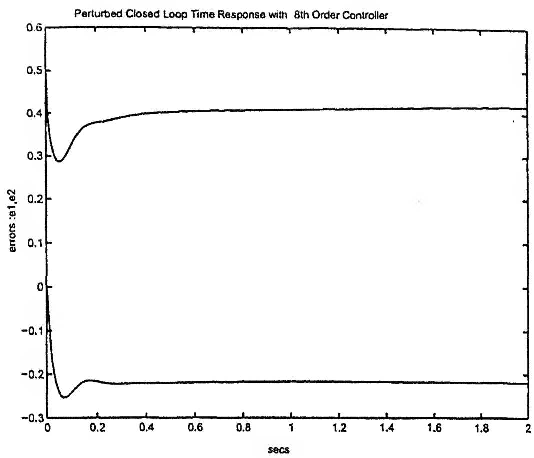

Figure 4.16: Step Response with 8th Order Controller for Perturbed System

4.3 Reduced Order Autopilot design for flexible Launch Vehicle

4.3.1 Introduction

This example consists of two parts. In section 4.3.2 a full order controller for the experimental data available for the launch vehicle is designed according to specifications. In the second part(section 4.3.3) we apply the proposed reduction algorithms to get reduced order controllers. The reduced order controllers thus achieved are found to meet all the performance specifications as guaranteed by Chapter 3.

4.3.2 Plant Description

The data for the launch vehicle is provided in terms of three (A, B, C, D) matrices representing the Nominal Plant (G_{nom}), the upper perturbed plant(G_u) and the lower perturbed plant(G_l). The upper and lower perturbed plants represent the extreme plants and all other plant descriptions are expected to lie between these two limits.

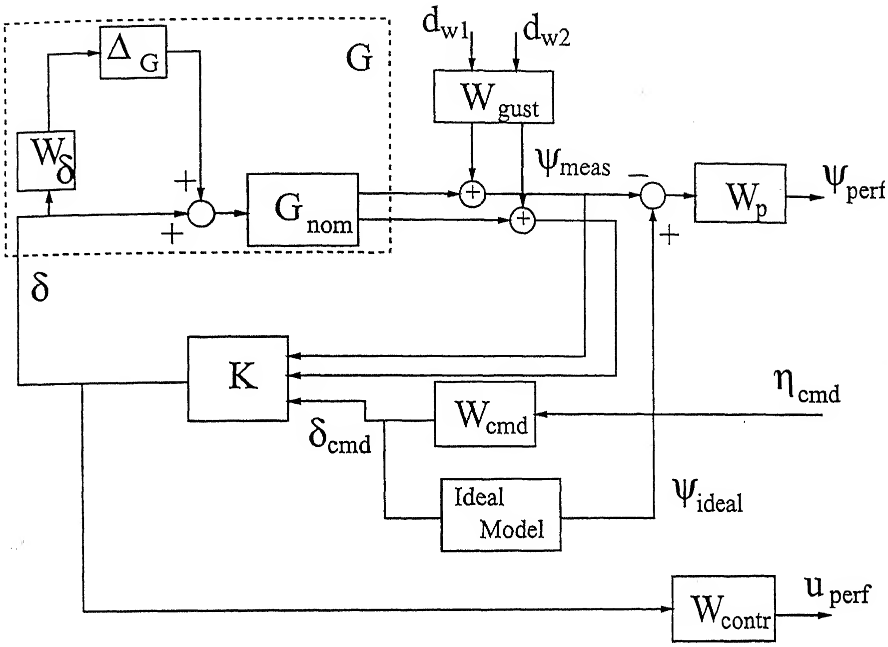

Figure 4.17: The Plant and Controller Interconnection with Weights

The (A, B, C, D) matrices incorporate the launch vehicle rigid body, slosh and launching modes along with actuators and the nozzle modes. The closed loop representation of the Plant and Controller interconnection is shown in figure 4.17. The dashed box named G represents the actual system consisting of the nominal model G_{nom} along with the uncertainty weight W_δ and the unstructured full block multiplicative uncertainty Δ_G. The input signals $[d_{w1}\ d_{w2}]$ along with W_{gust} represent the wing gust disturbance while $\delta_{cmd} = W_{cmd}\eta_{cmd}$ represent the reference signal. ψ_{perf} and u_{perf} are the performance variables created for converting the control objectives into $\mathcal{H}_\infty$ design objectives. The nominal plant G_{nom} consists of:

19 states

one control input

δ-angle of deflection

two measured outputs

ψ-attitude

$\dot{\psi}$-attitude rate

The control objective is that ψ_{meas} should track the command signal δ_{cmd}. In the $\mathcal{H}_\infty$ design perspective this objective is transformed into a transfer function minimization problem by suitable selection of weights. To convert the original control objectives into the $\mathcal{H}_\infty$ design objective we use a model following approach with the ideal model providing the required response(ψ_{ideal}) of the launch vehicle. The final design objective is thus minimization of the transfer function from $[\eta_{cmd}\, d_{w1}\, d_{w2}]^T$ to the performance signals $[\psi_{perf}\, u_{perf}]^T$.

Exogenous Disturbance, Commands: There are two sources of exogenous signals-wind gusts and angle of deflection commands.

Wind Gust: The Wind Gust data being unavailable, it is modelled as a direct low frequency disturbance on the controlled variable ψ_{meas}. The weights W_{gust} is chosen to be

$$W_{gust} = \begin{bmatrix} \frac{0.01(0.5s+1)}{(s+1)} & 0 \\ 0 & \frac{0.01(0.5s+1)}{(s+1)} \end{bmatrix}$$

This implies that the disturbance is expected to be within a bandwidth of 2 rad/s and the maximum amplitude expected is 1% of the command signal.

Deflection Command: The guidance system produces Deflection angle commands which the autopilot must make the launch vehicle to follow. Typical deflection commands are modelled as $\delta_{cmd} = W_{cmd}\eta_{cmd}$ where η_{cmd} is assumed to be any arbitrary signal with $\|\eta_{cmd}\|_\infty \leq 1$. Here the W_{cmd} is chosen as:

$$W_{cmd} = \frac{0.5(s+2)}{(s+1)}$$

The particular choice roughly implies that the command input is expected within a bandwidth of 2 rad/s. This being the bandwidth of the autopilot itself, this choice is justified.

Errors: Two separate variables must be kept small in the presence of the above disturbances. In this context these variables will be considered errors.

Control Input: This variable must be under 5 radians according to specifications. So the control weight is chosen as

$$W_{control} = \frac{1}{5}$$

The Attitude Error: The ideal attitude response to the angle of deflection command is chosen to be

$$\psi_{ideal} = \frac{4}{s^2 + 2 \times 0.7 \times 2s + 4}\delta_{cmd}$$

Thus our ideal plant has a natural frequency of 2 rad/s and a damping ratio of 0.7. The tracking error is defined as $\psi_{perf} = \psi_{ideal} - \psi_{meas}$. The tracking error signal is weighted by frequency depended weight W_p which is chosen as

$$W_p = \frac{0.5(s+1)^2}{(s+0.1)^2}$$

This weight emphasizes the need for low tracking error at low frequency. This restriction is relaxed slowly for higher frequencies. The actual gain and the exact form of the weight is arrived at iteratively and is somewhat arbitrary.

Uncertainty Model: The given data severely restricts any chance of non-conservative modelling of the uncertainty. As has already been discussed, the data is available as a nominal plant with upper and lower perturbed limits. Only way to model this sort of uncertainty is as unstructured full block multiplicative type. We describe the set of plants by

$$\mathcal{G} := \{G_{nom}(1 + \Delta_G W_{del}) : \Delta_G \ stable, \|\Delta_G\|_\infty < 1\}$$

where W_{del} is chosen as

$$W_{del} = \frac{5(s+1)^2}{(s+10)^2}$$

and Δ_G is full block type. The W_{del} chosen allows for small amount of uncertainty in the plant at low frequencies and the amount of allowed uncertainty increases smoothly at high frequencies. The particular choice of W_{del} ensures that any plant $G \in \mathcal{G}$ also belongs within the limits that has been set by the upper and lower perturbed plants.

Now that all the different weights have been specified we present a schematic diagram in figure 4.18 of the augmented plant in the generic feedback arrangement.

The augmented G_{AUG} consists of 28 states with 6 outputs and 5 inputs. The controller K is designed using D-K iteration with the "dkit" routine of MATLAB. The best K_f found is of 38 order with three inputs and one output. To check robust performance we define the augmented uncertainty block according to theorem 1.8.3.

$$\Delta_{AUG} = \left\{ \begin{bmatrix} \Delta_G & 0 \\ 0 & \Delta_p \end{bmatrix} : \Delta_G \in \mathbb{C}^{1\times1}, \Delta_p \in \mathbb{C}^{3\times2} \right\}$$

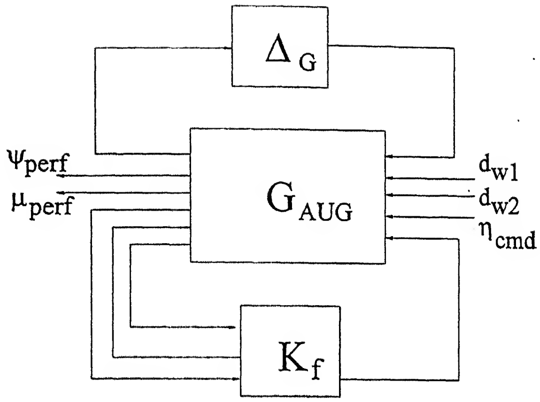

Figure 4.18: The Augmented Plant with Controller in the generic feedback arrangement

The upper and the lower bounds of $\mu(\mathcal{F}_l(G_{AUG}, K_f), \Delta_{AUG})$ are plotted in figure 4.19. The supremum over frequency of the upper bound is found to be 1.082, which is acceptable for our purposes. This 38^{th} order controller is found satisfactory in all respects as will be verified through various time simulations in the next section. But such high order controllers are hardly implementable, even on a computer. Thus it is a suitable candidate for testing the controller reduction algorithms of Chapter 3.

4.3.3 Reduced Order Controller design

It should be noted that both our reduction approaches $\mu A(P)$ and $\mu C(P)$ assumes the full order controller satisfying performance specifications to be known. We use K_f of previous section for this purpose. The results of the controller reduction are presented in table 4.3.

Figures 4.19, 4.20 and 4.21 show the μ plots for the full order and the reduced order controllers. All are found to be in the prescribed range, as had been guaranteed

Reduction Approach	Controller Order	$\mu(\mathcal{F}_l(P,K),\Delta)$
Full Order	38	1.082
$\mu A(P)$	16	1.2072
$\mu C(P)$	17	1.1713

Table 4.3: Results for Controller Reduction

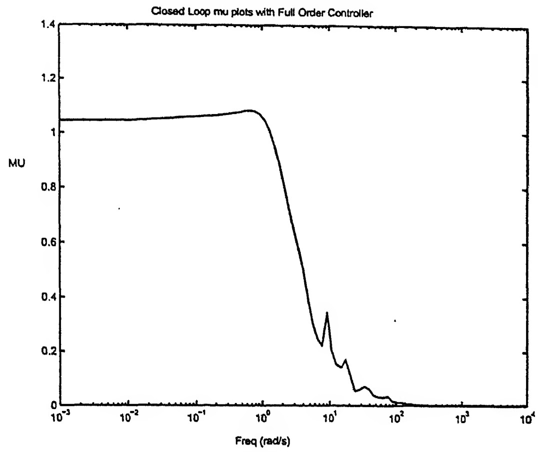

Figure 4.19: Closed Loop μ with Full Order Controller

a priori by proposition 3.2.2 and 3.3.1. The maximum singular value plots for the closed loop system with the different controllers are shown in figure 4.22. All three singular value plots lie below unity and thus indicate successful $\mathcal{H}_\infty$ design.

The various controllers in closed loop are now tested in time domain. Step responses to the tracking command are shown for the nominal and perturbed closed loop with the various full and reduced order controllers. The wind gust disturbance is modelled as a step input after 32 seconds. The full order step response is satisfactory. It not only gives a well-damped and tolerably quick response with a rise time of approximately 2 seconds and settling time of approximately 4 seconds, it also successfully rejects the wind gust disturbance. The peak overshoot is well within the

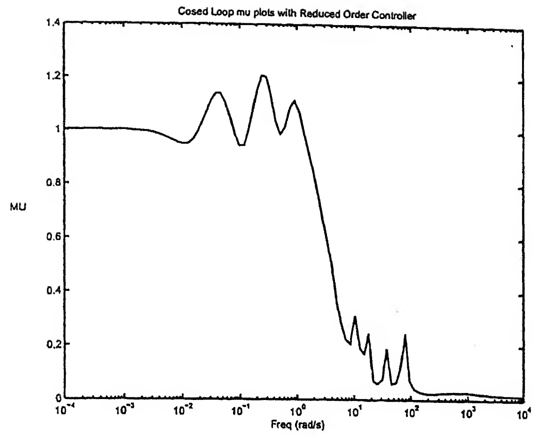

Figure 4.20: Closed Loop μ with Reduced Order($\mu A(P)$) Controller

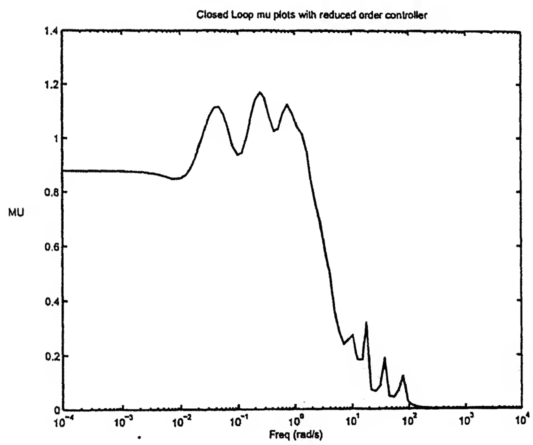

Figure 4.21: Closed Loop μ with Reduced Order($\mu C(P)$) Controller

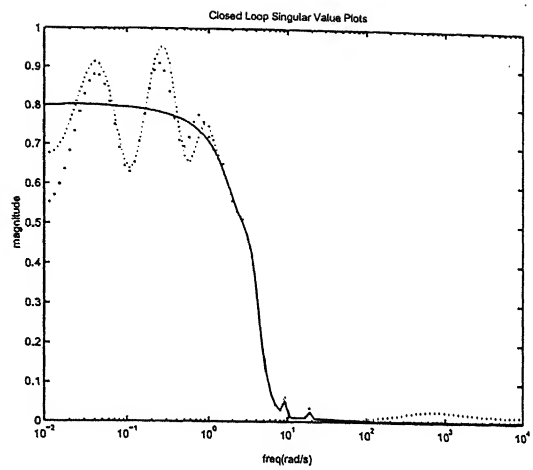

Figure 4.22: Closed Loop Singular Values:continuous-full order; dashed-$\mu A(P)$; dotted-$\mu C(P)$

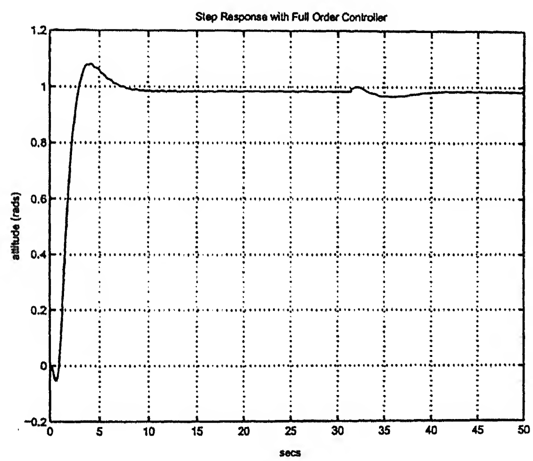

Figure 4.23: Step Response with Full Order Controller for Nominal Model

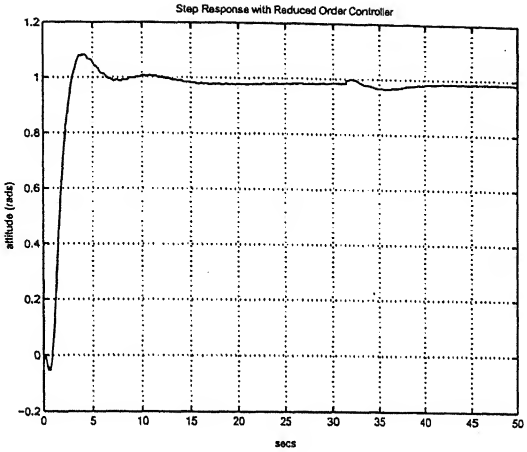

Figure 4.24: Step Response with $\mu A(P)$ Controller for Nominal Model

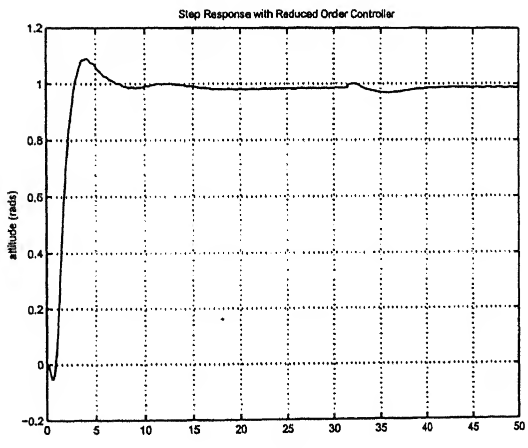

Figure 4.25: Step Response with $\mu C(P)$ Controller for Nominal Model

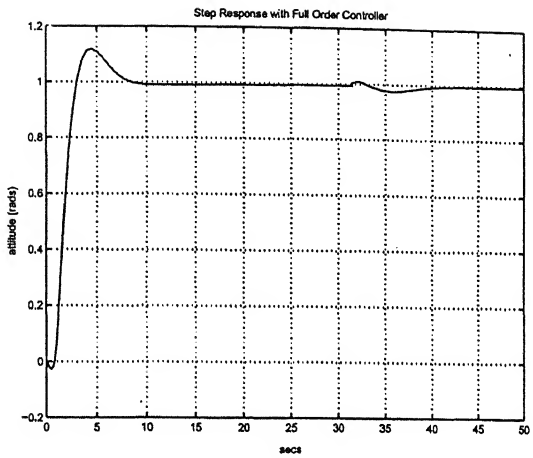

Figure 4.26: Step Response with Full Order Controller for Upper Perturbed Model

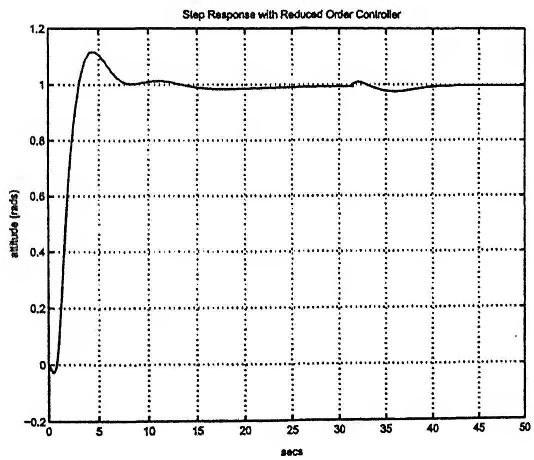

Figure 4.27: Step Response with $\mu A(P)$ Controller for Upper Perturbed Model

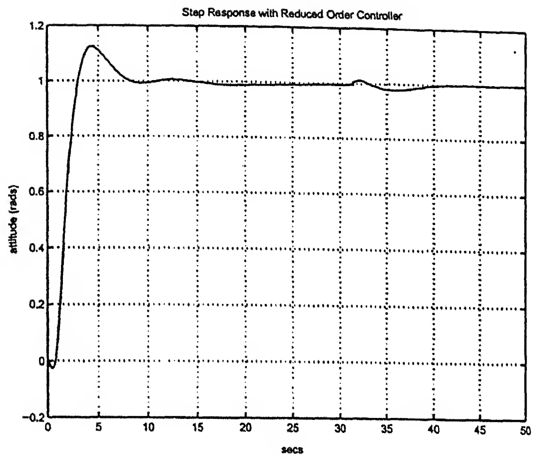

Figure 4.28: Step Response with $\mu C(P)$ Controller for Upper Perturbed Model

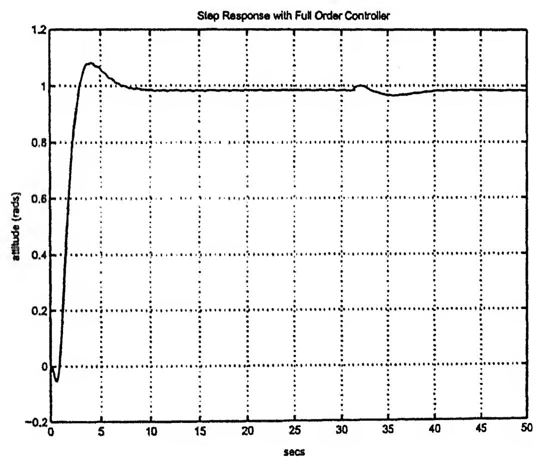

Figure 4.29: Step Response with Full Order Controller for Lower Perturbed Model

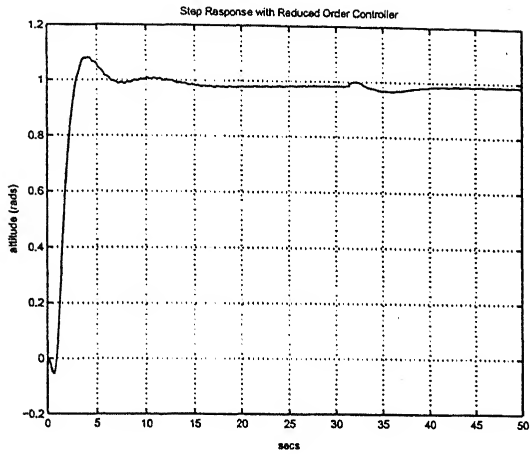

Figure 4.30: Step Response with $\mu A(P)$ Controller for Lower Perturbed Model

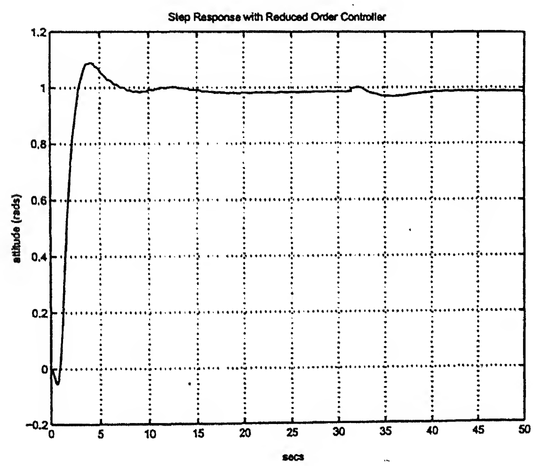

Figure 4.31: Step Response with $\mu C(P)$ Controller for Lower Perturbed Model

specified limit of 20%. The step responses with the reduced order controllers verify that the reduced order controllers perform as good as the full order controller. This has already been guaranteed by the corresponding μ plots and is borne out by the time responses. Thus the proposed controller reduction schemes successfully achieve considerable reduction in controller order (from 38 to 16 or 17) preserving robust performance and stability.

4.4 Conclusions

In this chapter the controller reduction techniques proposed in Chapter 3 has been tested on two practical examples. The full order design for the experimental HIMAT aircraft is taken from [8] and the proposed controller reduction algorithms are tested on this available design. The thirtieth order controller is successfully reduced to a fourteenth order controller without any optimization and this reduced order controller is found to have comparable performance with the full order controller. In the second part a complete μ-synthesis design is performed for the experimental data available from a flexible launch vehicle. The resulting 38th order controller is found to provide the desired characteristics. On application of the reduction algorithms a 15th order controller is found that performs comparably with the full order controller.

Chapter 5

Conclusions

In this final chapter we summarize by listing the contributions of this dissertation and recommending the directions for further research in this field.

5.1 Contributions of this dissertation

1. In this thesis a new proof has been given for the Kavranoğlu's additive reduction technique using the extension of the mixed μ theorem for uncertainties with frequency dependant bounds of Tits and Balakrishnan [25]. This proof provides a rigorous basis for additive controller reduction in the μ framework.

2. For the same technique the weights for two sided reduction, which are sufficient to guarantee the closed loop μ to remain less than unity, have been derived. It is also shown that these weights satisfy the desirable condition that their size is minimized in some sense.

3. One of the main drawbacks of the above method is that it assumes the controller to be approximated is stable. Robust controller design techniques like $\mathcal{H}_\infty$ loop shaping or μ-synthesis do not guarantee stability of the controllers. Thus a new coprime factor reduction scheme has been proposed, which do not require

stability of the full order controller but guarantees closed loop stability and performance with structured uncertainty. The coprime factor perturbations to the controller have been found to have a block diagonal structure thus improving the reduction algorithm.

4. The above algorithms have been tested on a widely studied benchmark HIMAT aircraft. The proposed algorithms have been found to work quite well producing more than 50% reduction in the controller order without optimization. Detailed simulation results have been presented with the full and the reduced order controllers for both the nominal and the perturbed systems.

5. A full μ synthesis design has been performed on the experimental data available from a flexible launch vehicle. The details of the design with the selected performance weights and the simulation results are presented. The reduction algorithms are also tested on this practical example and found to produce considerable reduction in the controller order while preserving performance.

5.2 Directions of Further Research

In the proposed methods, while the frequency dependance of the structured singular value μ has been utilized fully the spatial structure of μ have not been utilized. The structure of the frequency dependant set bounding the uncertainties used in Lemma 3.2.1 is very restrictive. The extension of the μ theorem for frequency dependent bounds on the uncertainty where the bounding set is more general, may lead to less conservative techniques of model reduction.

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Appendix

HIMAT Data

$$A = \begin{bmatrix} -0.0226 & -36.6000 & -18.9000 & -32.1000 \\ 0 & -1.9000 & 0.9830 & 0 \\ 0.0123 & -11.7000 & -2.6300 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ -0.4140 & 0 \\ -77.8000 & 22.4000 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 57.3000 & 0 & 0 \\ 0 & 0 & 0 & 57.3000 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The design weights are $W_\delta = \frac{50s+5000}{s+10000}$, $W_p = \frac{0.5s+0.9}{s+0.018}$ and $W_n = \frac{2s+2.56}{s+320}$.

Additive Reduction Weights

$$\gamma(s) = \frac{3.8918s^3 + 4.8483e+003s^2 + 1.5088e+006s + 2.7701e+003}{s^3 + 4.4616e+003s^2 + 4.9755e+006s + 1.8748e+003}$$

Multiplicative Reduction Weights

$$\gamma_{UV}(s) = \frac{0.7237s^3 + 458.3224s^2 + 7.2567e+004s + 169.8140}{s^3 + 4.1844e+003s^2 + 4.3773e+006s + 3.9744e+004}$$

Launch Vehicle Data

Additive Reduction Weights

$$\gamma(s) = \frac{9.6768s^3 + 2.5500e + 004s^2 + 9.4549e + 005s + 8.5589e + 006}{s^3 + 1.6302e + 003s^2 + 7.4495e + 005s + 1.0127e + 008}$$

Multiplicative Reduction Weights

$$\gamma_{UV}(s) = \frac{0.0866s^3 + 5.0172e + 003s^2 + 1.8747e + 005s + 1.0746e + 006}{s^3 + 3.6585e + 003s^2 + 8.2348e + 005s + 3.8928e + 007}$$